CISC 2210 – Introduction to Discrete Structures

Midterm 1 Exam Solutions

Mar 8, 2022

1. Let $D = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ be the set of all ten digits.

   (a) Define two proper subsets of D, $A \subset D$ and $B \subset D$, with the following properties:

   - The union of both sets is D: $A \cup B = D$.
   - The intersection of both sets contains two digits: $|A \cap B| = 2$.

   Example 1: $A = \{0, 1, 2\}$ and $B = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$: Both subsets are proper subsets of D because for example $3 \notin A$ and $0 \notin B$, the union $A \cup B = D$ contains all ten digits, and the size of the intersection $A \cap B = \{1, 2\}$ is 2.

   Example 2: $A = \{0, 1, 2, 3, 4, 5\}$ and $B = \{4, 5, 6, 7, 8, 9\}$: Both subsets are proper subsets of D because for example $6 \notin A$ and $3 \notin B$, the union $A \cup B = D$ contains all ten digits, and the size of the intersection $A \cap B = \{4, 5\}$ is 2.

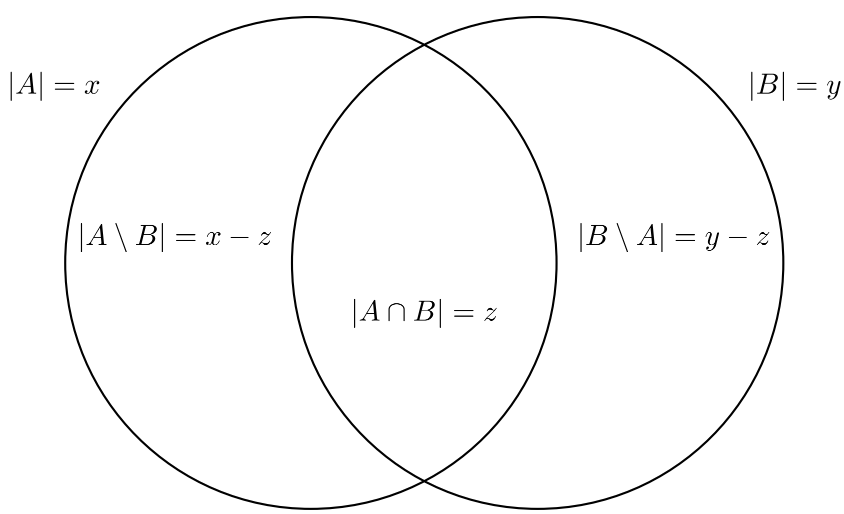
   Remark 1: Observe that since A and B are proper subsets of D both $A \setminus B$ and $B \setminus A$ cannot be empty because otherwise their union $A \cup B$ can not be D.

   Remark 2: Example 1 is one of the examples in which $|B| - |A| = 6$ which is the maximum possible size difference between the sets while Example 2 is one of the examples for which the two sets have the same size.

   (b) Prove that there are no two subsets of D, $A \subseteq D$ and $B \subseteq D$, with the following properties:

   - The intersection of both sets is not empty: $A \cap B \neq \emptyset$.
   - The sizes of A and B are different: $|A| \neq |B|$.
   - The sizes of $A \setminus B$ and $B \setminus A$ are the same: $|A \setminus B| = |B \setminus A|$.

   Proof: Consider any two subsets A and B of D. Assume that the size of A is x ($|A| = x$), the size of B is y ($|B| = y$), and the size of $A \cap B$ is z ($|A \cap B| = z$). The second assumption implies that $x \neq y$. Since the intersection of A and B is a subset of both A and B ($A \cap B \subseteq A$ and $A \cap B \subseteq B$), it follows that $z \leq x$ and $z \leq y$.
   Combining together the above three inequalities among x, y, and z implies that $x - z \neq y - z$. The proof follows since $x - z$ is the size of $A \setminus B$ ($|A \setminus B| = x - z$) and $y - z$ is the size of $B \setminus A$ ($|B \setminus A| = y - z$).



$$x \neq y \implies x - z \neq y - z$$

   Remark: The proof is correct for any two different size sets A and B even when $A \cap B = \emptyset$ because the proof did not use the assumptions that A and B are subsets of D and that $z \neq 0$.

2. Let A, B, and C, be three non-empty sets. Consider the following three sets:

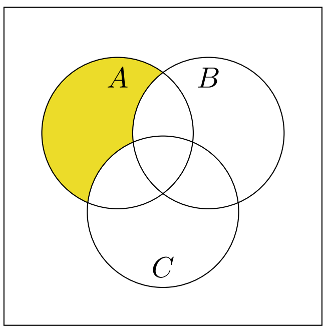
$$\begin{aligned} R &= (A \cap \overline{B} \cap \overline{C}) \cup (\overline{A} \cap B \cap \overline{C}) \cup (\overline{A} \cap \overline{B} \cap C) \\ S &= (A \cup B \cup C) \setminus (A \cap B \cap C) \\ T &= (A \setminus (B \cup C)) \cup (B \setminus (A \cup C)) \cup (C \setminus (A \cup B)) \end{aligned}$$

Which two of of these sets are identical? Why is the third set different from the two identical sets?

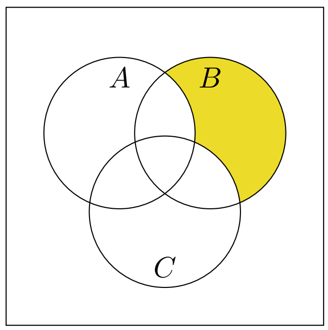
Proof that R $\equiv$ T: For any three sets X, Y, and Z the set $W = X \cap \overline{Y} \cap \overline{Z}$ is identical to the set $V = X \setminus (Y \cup Z)$. This is true because both W and V contain all the objects in X as long as they are not contained in Y and are not contained in Z.

As a result, $(A \cap \overline{B} \cap \overline{C}) \equiv (A \setminus (B \cup C))$, $(\overline{A} \cap B \cap \overline{C}) \equiv (B \setminus (A \cup C))$, and $(\overline{A} \cap \overline{B} \cap C) \equiv (C \setminus (A \cup B))$. This implies that $R \equiv T$.

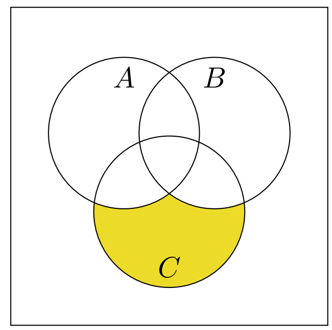
Proof that $S \not\equiv R$ and $S \not\equiv T$: All three sets R, S, and T cannot contain objects from $A \cap B \cap C$. However, S may contain objects from $A \cap B$ or $A \cap C$ or $B \cap C$ that are not contained in $A \cap B \cap C$. While both R and T cannot contain such objects.



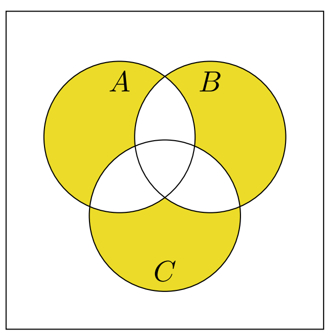
$A \cap \overline{B} \cap \overline{C} \equiv A \setminus (B \cup C)$



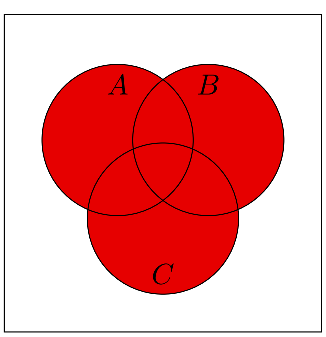
$\overline{A} \cap B \cap \overline{C} \equiv B \setminus (A \cup C)$



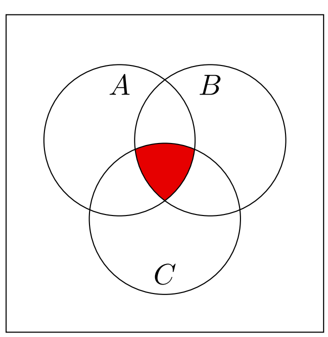
$\overline{A} \cap \overline{B} \cap C \equiv C \setminus (A \cup B)$



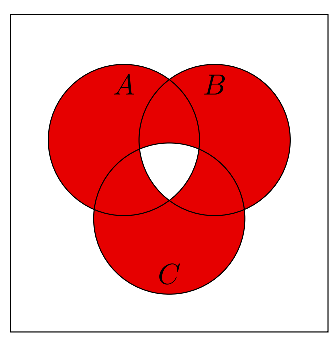
$R = (A \cap \overline{B} \cap \overline{C}) \cup (\overline{A} \cap B \cap \overline{C}) \cup (\overline{A} \cap \overline{B} \cap C) \quad \equiv \quad T = (A \setminus (B \cup C)) \cup (B \setminus (A \cup C)) \cup (C \setminus (A \cup B))$



$A \cup B \cup C$



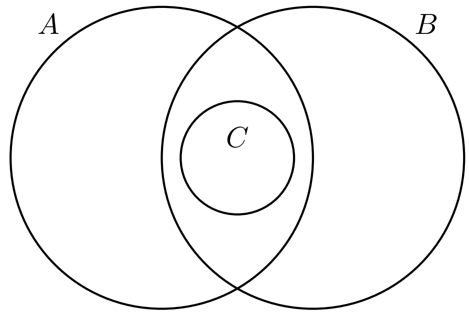
$A \cap B \cap C$



$S = (A \cup B \cup C) \setminus (A \cap B \cap C)$

3. Let A, B, and C be three non empty sets.

   (a) Draw the Venn-diagram for these sets in which C is a proper subset of the intersection between A and B ($C \subset (A \cap B)$).



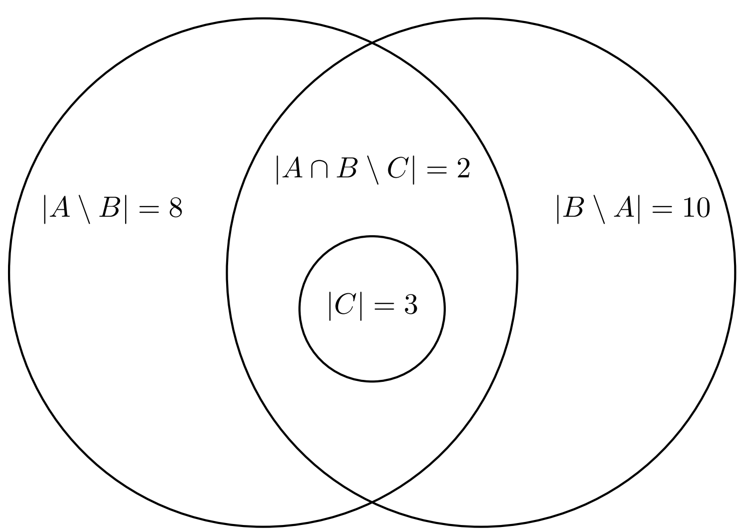
   (b) What are the sizes of A, B, and $A \cup B$ given the following data:
   - $|C| = 3$
   - $|A \cap B| = 5$
   - $|A \setminus C| = 10$
   - $|B \setminus C| = 12$

   Observation: $|X| = |X \setminus Y| + |X \cap Y|$ for any two sets X and Y because any object contained in X is either contained in $X \setminus Y$ or contained in $X \cap Y$ but not in both.

   Answer: $|\mathbf{A}| = \mathbf{13}$, $|\mathbf{B}| = \mathbf{15}$, and $|\mathbf{A} \cup \mathbf{B}| = \mathbf{23}$.

   Proof:
   - Since $|C| = 3$ and $|A \setminus C| = 10$ the above observation implies that
   $$|A| = |A \setminus C| + |C| = 10 + 3 = 13$$
   - Since $|C| = 3$ and $|B \setminus C| = 12$ the above observation implies that
   $$|B| = |B \setminus C| + |C| = 12 + 3 = 15$$
   - Since $|A \cap B| = 5$, the principle of inclusion exclusion implies that
   $$|A \cup B| = |A| + |B| - |A \cap B| = 13 + 15 - 5 = 23$$

   Remark: See below the sizes of the diagram's four disjoint zones: $(A \setminus B)$, $(B \setminus A)$, C, and $((A \cap B) \setminus C)$. Observe that indeed $|A \cap B| = 3+2 = 5$, $|A| = 8+2+3 = 13$, $|A \setminus C| = 13-3 = 10$, $|B| = 10+2+3 = 15$, $|B \setminus C| = 15 - 3 = 12$, and $|A \cup B| = 8 + 10 + 3 + 2 = 23$.



4. Let $S = \{0, 2, 4, 6, 8\}$ be the set of the five even digits and Let $T = \{1, 3, 5, 7, 9\}$ be the set of the five odd digits.

   For each one of the following expressions, determine if it is TRUE or FALSE.

   (a) $\forall_{x \in S} \forall_{y \in T} \, (y = x + 1)$
   The expression is FALSE because there exists at least one pair of numbers (x, y) for which x is even, y is odd, and $y \neq x + 1$. For example, $x = 0$ and $y = 9$.

   (b) $\exists_{x \in S} \exists_{y \in T} \, (y = x + 1)$

   The expression is TRUE because there exists at least one pair of numbers (x, y) for which x is even, y is odd, and $y = x + 1$. For example, $x = 4$ and $y = 5$.

   (c) $\forall_{x \in S} \exists_{y \in T} \, (y = x + 1)$

   The expression is TRUE because for any even number $x \in S$ there exists an odd number $y \in T$ such that $y = x + 1$: $1 = 0 + 1$, $3 = 2 + 1$, $5 = 4 + 1$, $7 = 6 + 1$, and $9 = 8 + 1$.

   (d) $\forall_{y \in T} \exists_{x \in S} \, (y = x + 1)$

   The expression is TRUE because for any odd number $y \in T$ there exists an even number $x \in S$ such that $y = x + 1$: $1 = 0 + 1$, $3 = 2 + 1$, $5 = 4 + 1$, $7 = 6 + 1$, and $9 = 8 + 1$,

   (e) $\exists_{x \in S} \forall_{y \in T} \, (y = x + 1)$

   The expression is FALSE because for any even number $x \in S$ there exists at least one odd number $y \in T$ such that $y \neq x + 1$. For example, $3 \neq 0 + 1$, $5 \neq 2 + 1$, $7 \neq 4 + 1$, $9 \neq 6 + 1$, and $1 \neq 8 + 1$.

   (f) $\exists_{y \in T} \forall_{x \in S} \, (y = x + 1)$

   The expression is FALSE because for any odd number $y \in T$ there exists at least one even number $x \in S$ such that $y \neq x + 1$. For example, $1 \neq 8 + 1$, $3 \neq 0 + 1$, $5 \neq 2 + 1$, $7 \neq 4 + 1$, and $9 \neq 6 + 1$.

5. The goal is to express the operator $\mathcal{XOR}$ with the operators $\mathcal{AND}$, $\mathcal{OR}$, and $\mathcal{NOT}$. The truth table for the operator $\mathcal{XOR}$ is:

x	y	$x \oplus y$
T	T	F
T	F	T
F	T	T
F	F	F

(a) Express $\mathcal{XOR}$ ($\oplus$) with $\mathcal{AND}$ ($\wedge$), $\mathcal{OR}$ ($\vee$), and $\mathcal{NOT}$ ($\neg$).

Solution 1: $x \oplus y$ is TRUE if and only if the truth assignments for x and y are different. Therefore, $x \oplus y = (x \wedge \neg y) \vee (\neg x \wedge y)$.

x	y	$x \wedge \neg y$	$\neg x \wedge y$	$(x \wedge \neg y) \vee (\neg x \wedge y)$
T	T	F	F	F
T	F	T	F	T
F	T	F	T	T
F	F	F	F	F

Solution 2: $x \oplus y$ is TRUE if and only if $x \vee y$ and $\neg x \vee \neg y$ are both TRUE because the truth assignment must assign TRUE to one of them and FALSE to the other one. Therefore, $x \oplus y = (x \vee y) \wedge (\neg x \vee \neg y)$.

x	y	$x \vee y$	$\neg x \vee \neg y$	$(x \vee y) \wedge (\neg x \vee \neg y)$
T	T	T	F	F
T	F	T	T	T
F	T	T	T	T
F	F	F	T	F

(b) Express $\mathcal{XOR}$ ($\oplus$) with $\mathcal{AND}$ ($\wedge$) and $\mathcal{NOT}$ ($\neg$). In this part, you cannot use $\mathcal{OR}$ ($\vee$).

De Morgan's law: $P \vee Q = \neg(\neg P \wedge \neg Q)$ for any two boolean expressions P and Q.

Solution 1: Apply the above De Morgan's law to get rid of the $\vee$ operator in the first solution of part (a). It follows that $x \oplus y = (x \wedge \neg y) \vee (\neg x \wedge y) = \neg(\neg(x \wedge \neg y) \wedge \neg(\neg x \wedge y))$.

x	y	$x \wedge \neg y$	$\neg x \wedge y$	$\neg(x \wedge \neg y)$	$\neg(\neg x \wedge y)$	$\neg(x \wedge \neg y) \wedge \neg(\neg x \wedge y)$	$\neg(\neg(x \wedge \neg y) \wedge \neg(\neg x \wedge y))$
T	T	F	F	T	T	T	F
T	F	T	F	F	T	F	T
F	T	F	T	T	F	F	T
F	F	F	F	T	T	T	F

Solution 2: Apply the above De Morgan's law to get rid of the two $\vee$ operators in the second solution of part (a). It follows that $x \oplus y = (x \vee y) \wedge (\neg x \vee \neg y) = \neg(\neg x \wedge \neg y) \wedge \neg(x \wedge y)$.

x	y	$\neg x \wedge \neg y$	$x \wedge y$	$\neg(\neg x \wedge \neg y)$	$\neg(x \wedge y)$	$\neg(\neg x \wedge \neg y) \wedge \neg(x \wedge y)$
T	T	F	T	T	F	F
T	F	F	F	T	T	T
F	T	F	F	T	T	T
F	F	T	F	F	T	F

6. You interrogate three people: A, B, and C. One of them stole your wallet.

- A claims that B stole the wallet.
- B claims that A stole the wallet.
- C agrees with B that A stole the wallet.

(a) Who stole the wallet if only one of them is lying while the other two are telling the truth?

Answer: A stole the wallet.

Proof 1: Since two out of the three people are telling the truth, it must be the case that these two people are B and C because they agree with each other. As a result, their statements that A stole the wallet is correct.

Proof 2: A and B contradict each other and therefore at least one of them is lying. Since two people are telling the truth, it follows that C is telling the truth and therefore C's statement that A stole the wallet is correct.

(b) Who stole the wallet if only one of them is telling the truth while the other two are lying?

Answer: B stole the wallet.

Proof 1: Since two out of the three people are lying, it must be the case that these two people are B and C because they agree with each other. As a result, A is telling the truth and its statement that B stole the wallet is correct.

Proof 2: A and B contradict each other and therefore at least one of them is telling the truth. Since two people are lying, it follows that C is lying and therefore B is also lying. Consequently, only A is telling the truth and therefore A's statement that B stole the wallet is correct.

Remark: Another way to solve the two parts of the problem is to examine all the three possibilities for who stole the wallet.

- If C stole the wallet then all three people are lying. Therefore, C cannot be the answer in both parts of the problem.
- If B stole the wallet then A is telling the truth and both B and C are lying. Therefore, B is the answer for part (b) of the problem.
- If A stole the wallet then B and C are telling the truth and A is lying. Therefore, A is the answer for part (a) of the problem.