CISC 2210 – Introduction to Discrete Structures

Midterm 1 Exam Solutions

Mar 8, 2022

1. Let $S = \{0, 2, 4, 6, 8\}$ be the set of the even digits and let $T = \{1, 3, 5, 7, 9\}$ be the set of the odd digits.

   (a) Define a subset A of the set of all digits with the following properties:

   - A contains at least one odd digit and at least one even digit: $A \cap S \neq \emptyset$ and $A \cap T \neq \emptyset$.
   - The intersection of A with S contains two more digits than the intersection of A with T: $|S \cap A| = |T \cap A| + 2$.

   Example 1: $A = \{0, 1, 2, 4\}$: The size of the non-empty set $A \cap S = \{0, 2, 4\}$ is 3 and the size of the non-empty set $A \cap T = \{1\}$ is 1. Indeed, $3 - 1 = 2$.

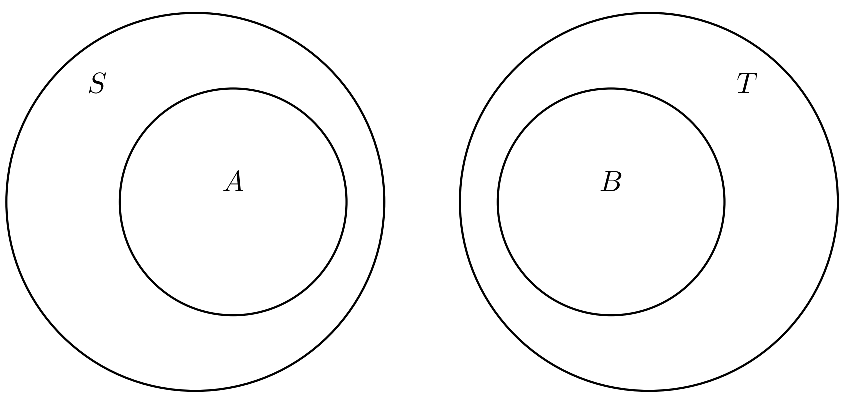
   Example 2: $A = \{0, 1, 2, 3, 4, 5, 6, 8\}$: The size of the non-empty set $A \cap S = \{0, 2, 4, 6, 8\}$ is 5 and the size of the non-empty set $A \cap T = \{1, 3, 5\}$ is 3. Indeed $5 - 3 = 2$.

   Remark: The set A in Example 1 is one of the smallest possible set ($|A| = 4$) that has the above two properties because its intersection with T must contain at least one digit and therefore its intersection with S must contain at least three digits for a total of at least four digits. On the other hand, the set in Example 2 is one of the largest possible set ($|A| = 8$) that has the above two properties because its intersection with S may contain at most five digits and therefore its intersection with T may contain at most three digits for a total of at most eight digits.

   (b) Prove that there are no two subsets $A \subseteq S$ and $B \subseteq T$ with the following properties:

   - Both sets are not empty: $A \neq \emptyset$ and $B \neq \emptyset$.
   - The intersection of A and B is not empty: $A \cap B \neq \emptyset$.

   Proof: S and T are disjoint sets because their intersection is empty: $S \cap T = \emptyset$ (a number is either odd or even). As a result, any two subsets A of S and B of T are disjoint as well: $A \cap B = \emptyset$.
   Suppose there exists $x \in A \cap B$. Then $x \in A$ and $x \in B$. Therefore, $x \in S$ and $x \in T$ which implies that $x \in S \cap T$. A contradiction.



$$T \cap S = \emptyset \implies A \cap B = \emptyset$$

2. Let A, B, and C, be three non-empty sets. Consider the following three sets:

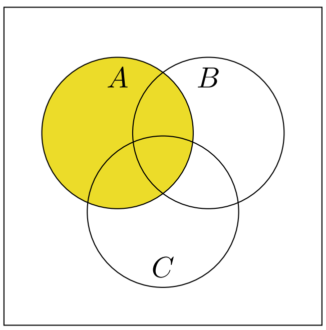
$$\begin{aligned} R &= A \cap (\overline{B \cap C}) \\ S &= (A \cup B) \cap (A \cup C) \\ T &= A \setminus (A \cap B \cap C) \end{aligned}$$

Which two of of these sets are identical? Why is the third set different from the two identical sets?

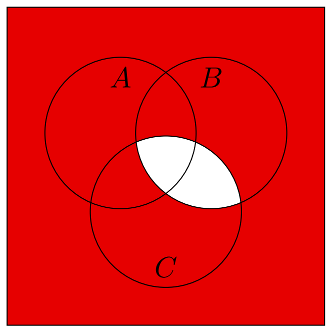
Proof that $\mathbf{R} \equiv \mathbf{T}$: Show that $x \in R$ if and only if $x \in T$.

- If $x \in R$ then $x \in A$ and $x \in (\overline{B \cap C})$. The latter implies that $x \notin (B \cap C)$. As a result, $x \notin A \cap (B \cap C)$. Consequently, $x \in T = A \setminus (A \cap B \cap C)$.
- If $x \in T$ then $x \in A$ but $x \notin (A \cap B \cap C)$. The latter implies that $x \notin (B \cap C)$. As a result, $x \in (\overline{B \cap C})$. Consequently, $x \in R = A \cap (\overline{B \cap C})$.

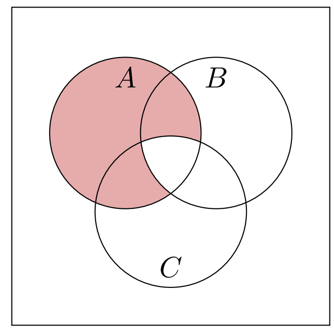
Proof that $S \not\equiv R$ and $S \not\equiv T$: S may contain objects from $A \cap B \cap C$ while both R and T cannot contain such objects. Also, S may contain objects from $B \cap C$ even if they are not contained in A while all objects of R and T must belong to A.



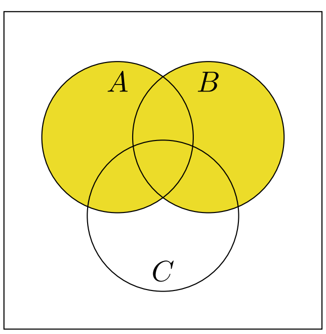
A



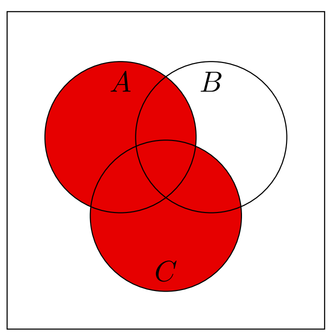
$\overline{B \cap C}$



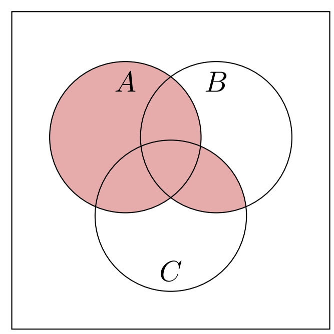
$R = A \cap (\overline{B \cap C})$



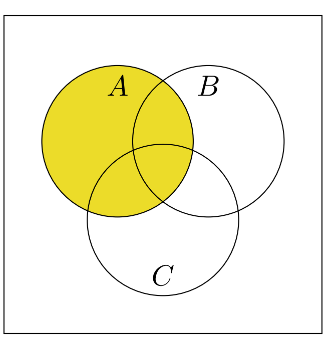
$A \cup B$



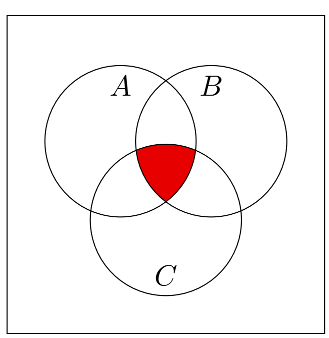
$A \cup C$



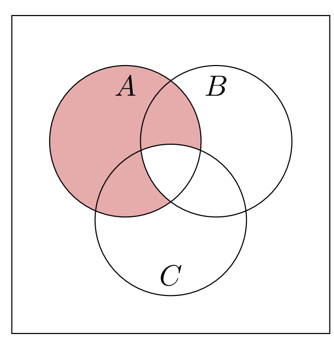
$S = (A \cup B) \cap (A \cup C)$



A



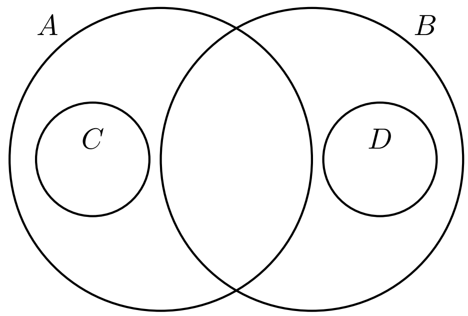
$A \cap B \cap C$



$T = A \setminus (A \cap B \cap C)$

3. Let A, B, C, and D be four non empty sets.

(a) Draw the Venn-diagram for these sets in which C is a proper subset of $A \setminus B$ ($C \subset (A \setminus B)$) and D is a proper subset of $B \setminus A$ ($D \subset (B \setminus A)$).



(b) What are the sizes of A, B, and $A \cap B$ given the following data:

- $|C| = 3$
- $|D| = 5$
- $|A \setminus B| = 10$
- $|B \setminus A| = 13$
- $|A \cup B| = 30$

Answer: $|\mathbf{A}| = \mathbf{17}$, $|\mathbf{B}| = \mathbf{20}$, and $|\mathbf{A} \cap \mathbf{B}| = \mathbf{7}$.

Proof: For any two sets X and Y the objects that are contained in the union $X \cup Y$ are either in X or in $Y \setminus X$ but not in both. Hence, $|X| = |X \cup Y| - |Y \setminus X|$. Therefore, for the sets A and B it follows that

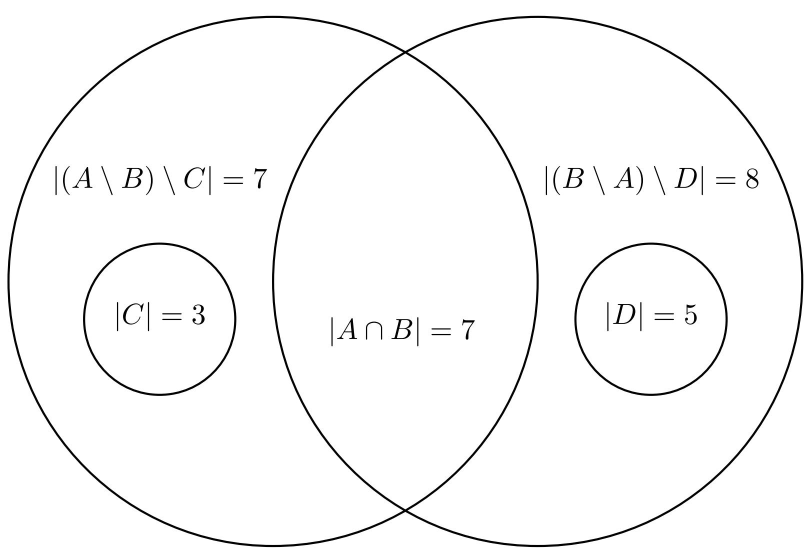
$$|A| = |A \cup B| - |B \setminus A| = 30 - 13 = 17$$
$$|B| = |B \cup A| - |A \setminus B| = 30 - 10 = 20$$

The principle of inclusion exclusion implies that $|A \cup B| = |A| + |B| - |A \cap B|$. Therefore,

$$|A \cap B| = |A| + |B| - |A \cup B| = 17 + 20 - 30 = 7$$

Remark 1: Observe that the sizes of C and D are irrelevant because C is a subset of $A \setminus B$ and D is a subset of $B \setminus A$.

Remark 2: See below the sizes of the diagram's five disjoint zones: $((A \setminus B) \setminus C)$, $((B \setminus A) \setminus D)$, C, D, and $(A \cap B)$. Observe that indeed $|A| = 7+3+7 = 17$, $|B| = 8+5+7 = 20$, and $|A \cup B| = 7+8+3+5+7 = 30$.



4. The following table summarizes which of the 5 colors Green, Magenta, Purple, Red, and Yellow are liked by the 5 students Alice, Bob, Carole, David, and Eva. If the value of an entry in the matrix is T, then the student from the entry's row likes the color from the entry's column, otherwise the student does not like the color.

For example, Bob likes Red because the value of the (Bob,Red) entry in the matrix is T while Eva does not like Yellow because the value of the (Eva,Yellow) entry in the matrix is F.

Student \ Colors	Green	Magenta	Purple	Red	Yellow
Alice	T	F	T	F	T
Bob	F	T	T	T	F
Carol	F	F	T	F	F
David	T	T	T	T	T
Eva	F	T	T	F	F

Let the set containing the five students be $S = \{\text{Alice}, \text{Bob}, \text{Carole}, \text{David}, \text{Eva}\}$ and let the set containing the five colors be $C = \{\text{Green}, \text{Magenta}, \text{Purple}, \text{Red}, \text{Yellow}\}$.

For each one of the following expressions, determine if it is TRUE or FALSE.

(a) $\forall_{x \in S} \forall_{y \in C} \, (x \text{ likes } y)$

The expression is FALSE because there is at least one F-entry in the matrix. For example, Alice does not like Magenta.

(b) $\exists_{x \in S} \exists_{y \in C} \, (x \text{ likes } y)$

The expression is TRUE because there is at least one T-entry in the matrix. For example, Carol likes Purple.

(c) $\forall_{x \in S} \exists_{y \in C} \, (x \text{ likes } y)$

The expression is TRUE because each row of the matrix contains at lease one T-entry. For example, Alice likes Green, Bob likes Red, Carol likes Purple, David likes Yellow, and Eva likes Magenta.

(d) $\forall_{y \in C} \exists_{x \in S} \, (x \text{ likes } y)$

The expression is TRUE because each column of the matrix contains at lease one T-entry. For example, Green is liked by Alice, Magenta is liked by Eva, Purple is liked by Carol, Red is liked by Bob, and Yellow is liked by David.

(e) $\exists_{x \in S} \forall_{y \in C} \, (x \text{ likes } y)$

The expression is TRUE because there exists a row such that all of its entries in the matrix are T-entries: David likes all the colors.

(f) $\exists_{y \in C} \forall_{x \in S} \, (x \text{ likes } y)$

The expression is TRUE because there exists a column such that all of its entries in the matrix are T-entries: Purple is liked by all the students.

5. The goal is to express the operator $\mathcal{NAND}$ with the operators $\mathcal{OR}$ and $\mathcal{NOT}$ and the operator $\mathcal{NOR}$ with the operators $\mathcal{AND}$ and $\mathcal{NOT}$. The truth tables for the operators $\mathcal{NAND}$ ($\uparrow$) and $\mathcal{NOR}$ ($\downarrow$) are:

x	y	$x \uparrow y$
T	T	F
T	F	T
F	T	T
F	F	T

x	y	$x \downarrow y$
T	T	F
T	F	F
F	T	F
F	F	T

(a) Express $\mathcal{NAND}$ ($\uparrow$) only with $\mathcal{OR}$ ($\vee$) and $\mathcal{NOT}$ ($\neg$). In this part, you cannot use $\mathcal{AND}$ ($\wedge$).

Answer: $\mathbf{x} \uparrow \mathbf{y} = \neg\mathbf{x} \vee \neg\mathbf{y}$

Proof: By definition $\mathcal{NAND}$ is $\mathcal{NOT}(\mathcal{AND})$. Therefore, $x \uparrow y = \neg(x \wedge y)$. Applying the De Morgan's low that $\neg(x \wedge y) = \neg x \vee \neg y$ implies that $x \uparrow y = \neg x \vee \neg y$.

x	y	$\neg x$	$\neg y$	$\neg x \vee \neg y$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

(b) Express $\mathcal{NOR}$ ($\downarrow$) only with $\mathcal{AND}$ ($\wedge$) and $\mathcal{NOT}$ ($\neg$). In this part, you cannot use $\mathcal{OR}$ ($\vee$).

Answer: $\mathbf{x} \downarrow \mathbf{y} = \neg\mathbf{x} \wedge \neg\mathbf{y}$

Proof: By definition $\mathcal{NOR}$ is $\mathcal{NOT}(\mathcal{OR})$. Therefore, $x \downarrow y = \neg(x \vee y)$. Applying the De Morgan's low that $\neg(x \vee y) = \neg x \wedge \neg y$ implies that $x \downarrow y = \neg x \wedge \neg y$.

x	y	$\neg x$	$\neg y$	$\neg x \wedge \neg y$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

6. You are visiting an island whose people are either always telling the truth (knights) or always lying (knaves). Explain why you cannot hear the following conversation between person A and person B:

   - A shouts at B that B is a knave.
   - B answers that A is right.

   Proof 1: If A is a knight, then A's statement that B is a knave is correct. But then B as a knave cannot agree with A. If A is a knave, then A's statement that B is a knave is wrong which implies that B is a knight. But as a knight B cannot agree with A. You cannot hear the above conversation since A must be either a knight or a knave.

   Proof 2: If B is a knight, then B cannot agree with A's statement that B is a knave which is wrong. If B is a knave, then B cannot agree with A's statement that B is a knave which is correct. You cannot hear the above conversation since B must be either a knight or a knave.

   Proof 3: A and B agree and therefore either both are knights or both are knaves. They cannot be knights since A claims that B is a knave and they cannot be knaves because both are telling the truth. You cannot hear the above conversation since A and B must be either knights or knaves.

   Proof 4: In the following truth table the first two columns list the four possible options of A and B being a knight or a knave. The last two columns represent the truth value of their statements based on the first two columns. Observe, that the third column agrees with the fourth column because B agrees with A.

A is a knight	B is a knight	A's statement	B's statement
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	T

   You cannot hear such a conversation because in none of the rows the entry in the third column is the same as the entry in the first column and the entry in the fourth column is the same as the entry in the second column.