CISC 2210 (TR11) – Introduction to Discrete Structures

Midterm 1 Exam

February 28, 2023

Id: ....

Problem	Maximum Points	Your Points
Sets 1	10	
Sets 2	15	
Sets 3	25	
Logic 1	15	
Logic 2	25	
Logic 3	10	
Total	100	

Structure, problem selection, and credit:

- You have 75 minutes to complete the exam.
- There are two parts: one for the topic of Sets and one for the topic of Logic. Each part contains three problems. See the chart above for the credit that you can earn for each of the six problems, for a total of 100 credits.
- You will get only partial credit if you fail to justify your answers. You will get 20% of the credit if you do not answer a problem. You will get zero credit for wrong answers.

Honor code: Students are expected to do this exam **by themselves** without any external help from other people, the Internet, books, or notes. Cheaters will be punished severely. At minimum, they will fail the exam, but they may fail the whole class. In addition, students who cheat risk disciplinary measures by Brooklyn College and CUNY.

1. Let $A \subseteq \mathcal{U}$ and $B \subseteq \mathcal{U}$ be two subsets of a universal set $\mathcal{U}$.

   Justify your answers to the following two questions.

   (a) What is $A \cup B$ if $\overline{A} \cap \overline{B} = \emptyset$?

   (b) What is $A \cap B$ if $\overline{A} \cup \overline{B} = \mathcal{U}$?

2. For a positive integer k, let $A_k = \{1, 2, \ldots, 2k\}$ be the set of all positive integers less or equal to $2k$.

   (a) Find the explicit form of each one of the sets A_1, A_2, and A_3.

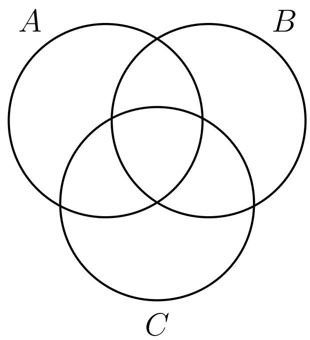
   (b) Find the explicit form of the set $S_3 = A_1 \cap A_2 \cap A_3$.

   (c) Find the explicit form of the set $T_3 = A_1 \cup A_2 \cup A_3$.

   (d) For a positive integer n, simplify $S_n = A_1 \cap A_2 \cap \cdots \cap A_n$. Justify your answer.

   (e) For a positive integer n, simplify $T_n = A_1 \cup A_2 \cup \cdots \cup A_n$. Justify your answer.

3. Let A, B, and C be three non-empty sets such that $(\mathbf{A} \cap \mathbf{B}) \subseteq \mathbf{C}$.



(a) In the above Venn-diagram mark the zone that MUST be empty given that $(A \cap B) \subseteq C$. Justify your answer.

(b) Given that $(A \cap B) \subseteq C$, the following additional data is known:

- A contains 8 objects ($|A| = 8$), B contains 10 objects ($|B| = 10$), and C contains 12 objects ($|C| = 12$).
- There are 4 objects that belong only to A ($|A \setminus (B \cup C)| = 4$), there are 5 objects that belong only to B ($|B \setminus (A \cup C)| = 5$), and there are 6 objects that belong only to C ($|C \setminus (A \cup B)| = 6$).

Justify your answers to the following two questions.

i. What is the size of the intersection of all three sets ($A \cap B \cap C$)?

ii. What is the size of the union of all three sets ($A \cup B \cup C$)?

4. A boolean formula is a **tautology** if its value is TRUE for any TRUE/FALSE assignment to its variables. A boolean formula is a **contradiction** if its value is FALSE for any TRUE/FALSE assignment to its variables.

   For each one of the following five statements, find out if it is TRUE or FALSE. Justify your answers.

   (a) Every boolean formula is either a tautology or a contradiction.

   (b) There exists a boolean formula that is neither a tautology nor a contradiction.

   (c) Finding one TRUE/FALSE assignment to the variables of a formula for which the value of the formula is TRUE is enough to prove that it is not a contradiction.

   (d) Finding one TRUE/FALSE assignment to the variables of a formula for which the value of the formula is FALSE is enough to prove that it is not a tautology.

   (e) If a formula is neither a tautology nor a contradiction then its value must be TRUE for exactly half of the TRUE/FALSE assignments to its variables.

5. The following is the truth table of the function $\mathcal{XOR}$ (denoted by $\oplus$) with the boolean variables x and y.

x	y	$x \oplus y$
T	T	F
T	F	T
F	T	T
F	F	F

(a) Let x, y, and z be three boolean variables. Prove that the following identity is correct:

$$((x \oplus y) \oplus z) \;=\; (y \oplus (x \oplus z))$$

(b) Let $x_1, x_2, \ldots, x_n$ be n ($n \geq 2$ could be very large) boolean variables. Consider the following boolean formula:

$$P = (x_1 \oplus x_2 \oplus \cdots \oplus x_n)$$

Justify your answers to the following four questions.

(i) What is the value of P (T or F) when all the n variables are FALSE?

(ii) What is the value of P (T or F) when all the n variables are TRUE?

Hint: The answer depends on whether n is even or odd.

(iii) What is the value of P (T or F) when out of the n variables exactly k ($0 \leq k \leq n$) are TRUE?

Hint: The answer depends on whether k is even or odd.

6. Alice, Bob, and Charlie are students at Brooklyn College. One is a CS major, one is an English major, and one is a Music major. The CS student always tells the truth, the English student always lies, and the Music student randomly sometimes lies or sometimes tells the truth.

   - Alice says: "Charlie studies English."
   - Bob says: "Alice studies CS."
   - Charlie says: "I study Music"

   Who studies CS, who studies English, and who studies Music?