CISC 2210 (TR11) – Introduction to Discrete Structures

Midterm 1 Exam — Solutions

February 28, 2023

1. Let $A \subseteq \mathcal{U}$ and $B \subseteq \mathcal{U}$ be two subsets of a universal set $\mathcal{U}$.

   (a) What is $A \cup B$ if $\overline{A} \cap \overline{B} = \emptyset$?

   Answer: $A \cup B = \mathcal{U}$.

   Proof 1: By one of the De Mogan's laws $\overline{A} \cap \overline{B} = \overline{A \cup B}$. As a result, $\overline{A \cup B} = \emptyset$. After negating both sides of this equality, since $\overline{\emptyset} = \mathcal{U}$, it follows that $A \cup B = \mathcal{U}$.

   Proof 2: Let $x \in \mathcal{U}$ be an arbitrary object in the universal set $\mathcal{U}$. Since $\overline{A} \cap \overline{B} = \emptyset$, it follows that either $x \in \overline{\overline{A}} = A$ or $x \in \overline{\overline{B}} = B$. This implies that $x \in A \cup B$. As a result, all objects of $\mathcal{U}$ belong to $A \cup B$ and therefore $A \cup B = \mathcal{U}$.

   (b) What is $A \cap B$ if $\overline{A} \cup \overline{B} = \mathcal{U}$?

   Answer: $A \cap B = \emptyset$.

   Proof 1: By one of the De Mogan's laws $\overline{A} \cup \overline{B} = \overline{A \cap B}$. As a result, $\overline{A \cap B} = \mathcal{U}$. After negating both sides of this equality, since $\overline{\mathcal{U}} = \emptyset$, it follows that $A \cap B = \emptyset$.

   Proof 2: Assume that there exists $x \in \mathcal{U}$ such that $x \in A \cap B$. This means that $x \in A$ and $x \in B$ and therefore $x \notin \overline{A}$ and $x \notin \overline{B}$. This implies that $x \notin \overline{A} \cup \overline{B}$. This is a contradiction to the assumption that $\overline{A} \cup \overline{B} = \mathcal{U}$. Therefore, $A \cap B$ must be the empty set.

2. For a positive integer k, let $A_k = \{1, 2, \ldots, 2k\}$ be the set of all positive integers less or equal to $2k$.

   (a) Find the explicit form of each one of the sets A_1, A_2, and A_3.

   Answer: $A_1 = \{1, 2\}$, $A_2 = \{1, 2, 3, 4\}$, and $A_3 = \{1, 2, 3, 4, 5, 6\}$

   (b) Find the explicit form of the set $S_3 = A_1 \cap A_2 \cap A_3$.

   Answer: $S_3 = \{1, 2\} = A_1$ because $A_1 \subset A_2 \subset A_3$.

   (c) Find the explicit form of the set $T_3 = A_1 \cup A_2 \cup A_3$.

   Answer: $T_3 = \{1, 2, 3, 4, 5, 6\} = A_3$ because $A_1 \subset A_2 \subset A_3$.

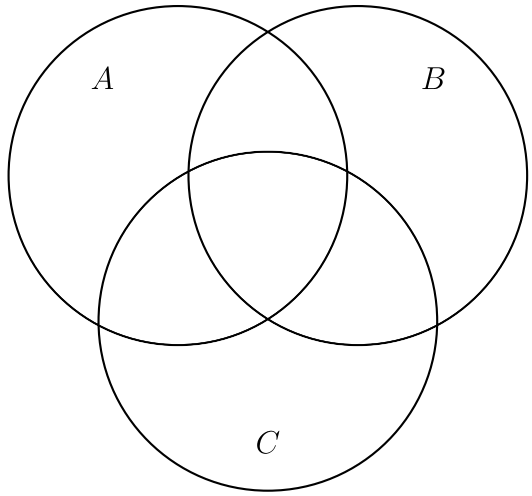
   (d) For a positive integer n, simplify $S_n = A_1 \cap A_2 \cap \cdots \cap A_n$.

   Answer: $S_n = \{1, 2\} = A_1$ because $A_1 \subset A_2 \subset \cdots \subset A_n$.

   (e) For a positive integer n, simplify $T_n = A_1 \cup A_2 \cup \cdots \cup A_n$.

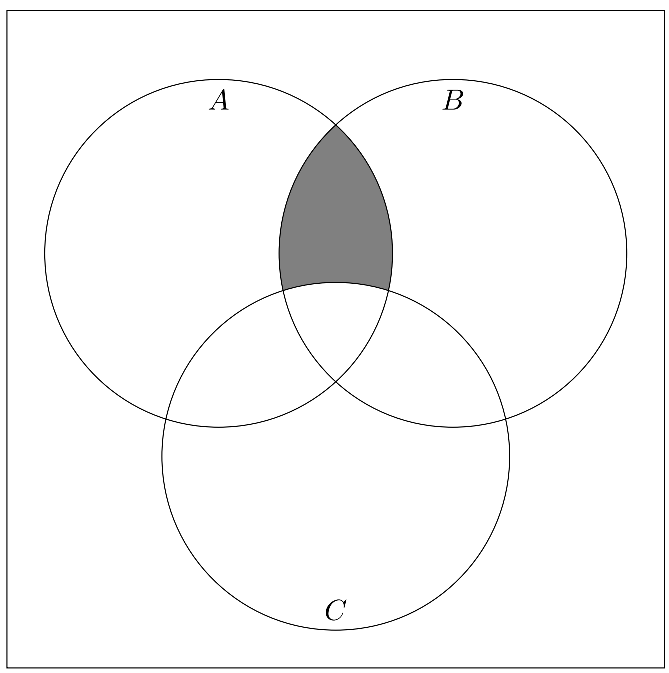
   Answer: $T_n = \{1, 2, \ldots, 2n\} = A_n$ because $A_1 \subset A_2 \subset \cdots \subset A_n$.

3. Let A, B, and C be three non-empty sets such that $(\mathbf{A} \cap \mathbf{B}) \subseteq \mathbf{C}$.



(a) In the above Venn-diagram mark the zone that MUST be empty given that $(A \cap B) \subseteq C$.

Answer: The zone $(A \cap B) \setminus C = AB\overline{C}$ must be empty because no object may belong to both A and B if it does not belong to C.



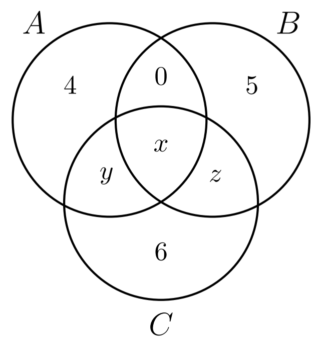
(b) Given that $(A \cap B) \subseteq C$, the following additional data is known:

- A contains 8 objects ($|A| = 8$), B contains 10 objects ($|B| = 10$), and C contains 12 objects ($|C| = 12$).
- There are 4 objects that belong only to A ($|A \setminus (B \cup C)| = 4$), there are 5 objects that belong only to B ($|B \setminus (A \cup C)| = 5$), and there are 6 objects that belong only to C ($|C \setminus (A \cup B)| = 6$).

i. What is the size of the intersection of all three sets ($A \cap B \cap C$)?

ii. What is the size of the union of all three sets ($A \cup B \cup C$)?

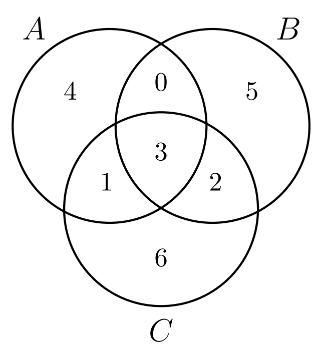
Answer: The diagram below reflects the data given in the second bullet and the answer from part (a). Denote by $x = |A \cap B \cap C|$, by $y = |(A \cap C) \setminus B|$, and by $z = |(B \cap C) \setminus A|$ the sizes of the zones for which the data was not explicit about their sizes.



The given size of A implies that $8 = 4 + 0 + x + y$, the given size of B implies that $10 = 5 + 0 + x + z$, and the given size of C implies that $12 = 6 + x + y + z$. All together, there are three equations with three variables

$$\begin{aligned} x + y &= 4 & (1) \\ x + z &= 5 & (2) \\ x + y + z &= 6 & (3) \end{aligned}$$

Equations (1) and (3) imply that $z = 2$ while equations (2) and (3) imply that $y = 1$. These values of y and z and equation (3) imply that $x = 3$. See the diagram below for the sizes of all the seven zones in the union of all three sets.



Final answers to parts (i) and (ii):

i. The size of the intersection of all three sets is $\mathbf{3} = |A \cap B \cap C|$.

ii. The size of the union of all three sets is $\mathbf{21} = 6 + 5 + 4 + 3 + 2 + 1 = |A \cup B \cup C|$.

4. A boolean formula is a **tautology** if its value is TRUE for any TRUE/FALSE assignment to its variables. A boolean formula is a **contradiction** if its value is FALSE for any TRUE/FALSE assignment to its variables.

   For each one of the following five statements, find out if it is TRUE or FALSE.

   (a) Every boolean formula is either a tautology or a contradiction.

   Answer: TRUE only for formulas with one variables. For formulas with more than one variable the answer is FALSE.

   In fact, most of the formulas are neither tautologies nor contradictions. For example, the all AND and all OR formulas.

   (b) There exists a boolean formula that is neither a tautology nor a contradiction.

   Answer: FALSE only for formulas with one variables. For formulas with more than one variable the answer is TRUE.

   In fact, most of the formulas are neither tautologies nor contradictions. For example, the all AND and all OR formulas.

   (c) Finding one TRUE/FALSE assignment to the variables of a formula for which the value of the formula is TRUE is enough to prove that it is not a contradiction.

   Answer: TRUE because then not all the TRUE/FALSE assignments imply a FALSE value to the boolean formula as required by the definition of a contradiction.

   (d) Finding one TRUE/FALSE assignment to the variables of a formula for which the value of the formula is FALSE is enough to prove that it is not a tautology.

   Answer: TRUE because then not all the TRUE/FALSE assignments imply a TRUE value to the boolean formula as required by the definition of a tautology.

   (e) If a formula is neither a tautology nor a contradiction then its value must be TRUE for exactly half of the TRUE/FALSE assignments to its variables.

   Answer: FALSE. For example, the all AND formula has only one assignment that satisfies it and the all OR formula has only one assignment that does not satisfy it.

5. The following is the truth table of the function $\mathcal{XOR}$ (denoted by $\oplus$) with the boolean variables x and y.

x	y	$x \oplus y$
T	T	F
T	F	T
F	T	T
F	F	F

(a) Let x, y, and z be three boolean variables. Prove that the following identity is correct:

$$((x \oplus y) \oplus z) \;=\; (y \oplus (x \oplus z))$$

Proof: The left table below is the truth table of the left side of the identity while the right table below is the truth table of the right side of the identity. The identity is correct because the last columns of both tables are identical.

x	y	z	$x \oplus y$	$(x \oplus y) \oplus z$
T	T	T	F	T
T	T	F	F	F
T	F	T	T	F
T	F	F	T	T
F	T	T	T	F
F	T	F	T	T
F	F	T	F	T
F	F	F	F	F

x	y	z	$x \oplus z$	$y \oplus (x \oplus z)$
T	T	T	F	T
T	T	F	T	F
T	F	T	F	F
T	F	F	T	T
F	T	T	T	F
F	T	F	F	T
F	F	T	T	T
F	F	F	F	F

Remark: The function $\mathcal{XOR}$ is commutative and associative. Therefore, for any integer n, any order for evaluating the formula $(x_1 \oplus x_2 \oplus \cdots \oplus x_n)$ with the n variables $x_1, x_2, \ldots, x_n$ will yield the same result.

The truth tables above demonstrates this for two out of the three possible orders to evaluate the formula $(x \oplus y \oplus z)$. In the left side of the identity the first pair of variables to be evaluated is (x, y) and in the right side of the identity the first pair of variables to be evaluated is (x, z). The third possible way is when the pair of variables (y, z) is evaluated first as is the case in the formulas $((y \oplus z) \oplus x)$ and $(x \oplus (y \oplus z))$

(b) Let $x_1, x_2, \ldots, x_n$ be n ($n \geq 2$) boolean variables. Consider the following boolean formula:

$$P = (x_1 \oplus x_2 \oplus \cdots \oplus x_n)$$

Notations: For $1 \leq i \leq n$, let $P_i = (x_1 \oplus x_2 \oplus \cdots \oplus x_i)$. In particular, $P_1 = x_1$ and $P_n = P$.

Evaluating P: It is done from left to right. As a result, the evaluation process is as follows

$$\begin{array}{rcllcl}
P_1 & = & (x_1) & & = & (x_1) \\
P_2 & = & (x_1 \oplus x_2) & & = & (P_1 \oplus x_2) \\
P_3 & = & (x_1 \oplus x_2 \oplus x_3) & & = & (P_2 \oplus x_3) \\
 & \vdots & & & & \\
P_i & = & (x_1 \oplus x_2 \oplus \cdots \oplus x_i) & & = & (P_{i-1} \oplus x_i) \\
 & \vdots & & & & \\
P = P_n & = & (x_1 \oplus x_2 \oplus \cdots \oplus x_{n-1} \oplus x_n) & & = & (P_{n-1} \oplus x_n)
\end{array}$$

i. What is the value of P (TRUE or FALSE) when all the n variables are FALSE?

Answer: P is FALSE.

Proof: Since $x_1 = F$, it follows by definition that $P_1 = F$. Next, $P_2 = (P_1 \oplus x_2) = F$ because $(F \oplus F) = F$. Next, $P_3 = (P_2 \oplus x_3) = F$ because $(F \oplus F) = F$. Continue this way to show that $P_4 = P_5 = \cdots = P_n = F$.
The answer follows because in particular $P = P_n$ is FALSE.

ii. What is the value of P (TRUE or FALSE) when all the n variables are TRUE?

Answer: P is TRUE when n is odd and P is FALSE when n is even.

Proof sketch: Since $x_1 = T$, it follows that $P_1 = T$. Next, $P_2 = (P_1 \oplus x_2) = F$ because $(T \oplus T) = F$. Next, $P_3 = (P_2 \oplus x_3) = T$ because $(F \oplus T) = T$.
In general, the value of P_i alternates from T to F and then back to T depending on the parity of i. If i is odd then $P_i = T$ and if i is even then $P_i = F$.
As a result, $P = P_n$ is TRUE for an odd n while $P = P_n$ is FALSE for an even n.

iii. What is the value of P (TRUE or FALSE) when out of the n variables exactly k ($0 \leq k \leq n$) are TRUE?

Answer: P is TRUE when k is odd and P is FALSE when k is even.

Proof sketch: Observe that $P_i = (P_{i-1} \oplus x_i)$ is different than P_{i-1} only if $x_i = T$. This is because $(R \oplus F) = R$ while $(R \oplus T) = \neg R$ for either $R = T$ or $R = F$.
As a result, the value of $(x_1 \oplus x_2 \oplus \cdots \oplus x_n)$ is the same as the value of $(x_{i_1} \oplus x_{i_2} \oplus \cdots \oplus x_{i_k})$ where $1 \leq i_1, i_2, \ldots, i_k \leq n$ are all the indices i for which $x_i = T$. This is because by the observation, all the other x_i (those for which $x_i = F$) do not change the value of P_{i-1}.
The answer is a corollary of the result from part (ii).

6. Alice, Bob, and Charlie are students at Brooklyn College. One is a CS major, one is an English major, and one is a Music major. The CS student always tells the truth, the English student always lies, and the Music student randomly sometimes lies or sometimes tells the truth.
   - Alice says: "Charlie studies English."
   - Bob says: "Alice studies CS."
   - Charlie says: "I study Music"

   Who studies CS, who studies English, and who studies Music?

   Answer: Alice studies CS, Bob studies Music, and Charlie studies English.

   Proof I: Using a process of elimination, identify the CS major, who must always tell the truth:
   - Charlie cannot be the CS major. If he were, his statement "I study Music" would have to be true. However, a student cannot be both the CS major and the Music major. This creates a direct contradiction.
   - Bob cannot be the CS major. If he were, his statement "Alice studies C" would have to be true. This would mean Alice is the CS major, but that is impossible if Bob is. This also creates a contradiction, as two people cannot have the same major.
   - Therefore, Alice must be the CS major. Since the other two possibilities lead to contradictions, Alice is the CS major by elimination.

   Now, verify this conclusion:
   - As the CS major, Alice tells the truth. Her statement, "Charlie studies English," must be true.
   - This confirms that Charlie is the English major. As the English major, Charlie must always lie. His statement, "I study Music" is indeed false, which is consistent with the rules.
   - By elimination, Bob must be the Music major. The Music major's statements can be either true or false. Bob said, "Alice studies CS" which in this case is a true statement. This is perfectly acceptable for the Music major.

   Every statement aligns perfectly within this single, contradiction-free scenario.

   Proof II: Test the initial assumption of whether Bob is lying or telling the truth. Start with the assumption that Bob is lying:
   - If Bob is lying, he cannot be the CS major, as the CS major always tells the truth.
   - Bob's statement is, "Alice studies CS." Since it is assumed he is lying, the opposite must be true: 'Alice does not study CS."
   - With both Bob and Alice eliminated as the CS major, Charlie must be the CS major by default.
   - As the CS major, Charlie must tell the truth. However, Charlie's statement is, "I study Music" This would mean he is the Music major, which contradicts our conclusion that he is the CS major.
   - This entire scenario leads to a logical contradiction. Therefore, the initial assumption that Bob is lying must be false.

   The only remaining possibility is that Bob is telling the truth:
   - Since Bob is telling the truth, his statement—"Alice studies CS"—must be true. This means Alice is the CS major.
   - As the CS major, Alice must also be telling the truth. Her statement is, "Charlie studies English" which confirms that Charlie is the English major.
   - As the English major, Charlie must always lie. His statement, "I study Music," is indeed a lie, which perfectly aligns with the rules.
   - The only remaining major for Bob is Music. His truth-telling is consistent with the random nature of the Music major.

   This conclusion is logical and free of contradictions.