CISC 2210 (TR2) – Introduction to Discrete Structures

Midterm 1 Exam

February 28, 2023

Id: ....

Problem	Maximum Points	Your Points
Sets 1	10	
Sets 2	15	
Sets 3	25	
Logic 1	15	
Logic 2	25	
Logic 3	10	
Total	100	

Structure, problem selection, and credit:

- You have 75 minutes to complete the exam.
- There are two parts: one for the topic of Sets and one for the topic of Logic. Each part contains three problems. See the chart above for the credit that you can earn for each of the six problems, for a total of 100 credits.
- You will get only partial credit if you fail to justify your answers. You will get 20% of the credit if you do not answer a problem. You will get zero credit for wrong answers.

Honor code: Students are expected to do this exam **by themselves** without any external help from other people, the Internet, books, or notes. Cheaters will be punished severely. At minimum, they will fail the exam, but they may fail the whole class. In addition, students who cheat risk disciplinary measures by Brooklyn College and CUNY.

1. The cardinality of a set A is its size and is denoted by $|A|$.

   Justify your answers to the following two questions.

   (a) Let A and B be two sets. Assume that

   $$|A \setminus B| = |B \setminus A|$$

   What is the relationship between the cardinalities of A and B?

   (b) Let S and T be two sets. Suppose that the cardinality of the union of the two sets S and T is larger by exactly one than the cardinality of the intersection of S and T:

   $$|S \cup T| = |S \cap T| + 1$$

   What is the relationship between the sets S and T?

2. For a positive integer k, let $A_k = \{-k, -(k-1), \ldots, -1, 0, 1, \ldots, (k-1), k\}$ be the set of all integers between $-k$ and k.

(a) Find the explicit form of each one of the sets A_1, A_2, and A_3.

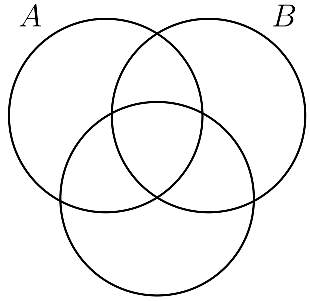
(b) Find the explicit form of the set $S_3 = A_1 \cap A_2 \cap A_3$.

(c) Find the explicit form of the set $T_3 = A_1 \cup A_2 \cup A_3$.

(d) For a positive integer n, simplify $S_n = A_1 \cap A_2 \cap \cdots \cap A_n$. Justify your answer.

(e) For a positive integer n, simplify $T_n = A_1 \cup A_2 \cup \cdots \cup A_n$. Justify your answer.

3. Let A, B, and C be three non-empty sets such that $\mathbf{C} \subseteq (\mathbf{A} \cup \mathbf{B})$.



(a) In the above Venn-diagram mark the zone that MUST be empty given that $C \subseteq (A \cup B)$. Justify your answer.

(b) Given that $C \subseteq (A \cup B)$, the following additional data is known:

- A contains 14 objects ($|A| = 14$), B contains 12 objects ($|B| = 12$), and C contains 9 objects ($|C| = 9$).
- There are 6 objects that belong only to A ($|A \setminus (B \cup C)| = 6$), there are 5 objects that belong only to B ($|B \setminus (A \cup C)| = 5$), and there is only one object in the intersection of A and B that is not in C ($|(A \cap B) \setminus C| = 1$).

Justify your answers to the following two questions.

i. What is the size of the intersection of all three sets ($A \cap B \cap C$)?

ii. What is the size of the union of all three sets ($A \cup B \cup C$)?

4. Let x, y, and z be three boolean variables.

   Justify your answers to the following three problems.

   (a) Define a boolean formula that must contain x, y, and z that is satisfied by **exactly one** TRUE/FALSE assignment to x, y, and z.

   (b) Define a boolean formula that must contain x, y, and z that is satisfied by **exactly seven** TRUE/FALSE assignments to x, y, and z.

   (c) Define a boolean formula that must contain x, y, and z for which **exactly four** TRUE/FALSE assignments to x, y, and z satisfy the formula.

5. The following is the truth table of the function $\mathcal{EQUIV}$ (denoted by $\equiv$) with the boolean variables x and y.

x	y	$x \equiv y$
T	T	T
T	F	F
F	T	F
F	F	T

(a) Let x, y, and z be three boolean variables. Prove that the following identity is correct:

$$((x \equiv y) \equiv z) \;=\; (y \equiv (x \equiv z))$$

(b) Let $x_1, x_2, \ldots, x_n$ be n ($n \geq 2$ could be very large) boolean variables. Consider the following boolean formula:

$$P \;=\; (x_1 \equiv x_2 \equiv \cdots \equiv x_n)$$

Justify your answers to the following three questions.

(i) What is the value of P (T or F) when all the n variables are TRUE?

(ii) What is the value of P (T or F) when all the n variables are FALSE?

Hint: The answer depends on whether n is even or odd.

(iii) What is the value of P (T or F) when out of the n variables exactly k ($0 \leq k \leq n$) are FALSE?

Hint: The answer depends on whether k is even or odd.

6. Only one of Alice, Bob or Charlie failed the test while the other two got an A on the test.
   - Alice says that Bob failed.
   - Bob says that Alice is lying.
   - Charlie says that he did not fail.

   Only one of them is telling the truth.

   Who failed the test and who is the one telling the truth?