CISC 2210 (TR2) – Introduction to Discrete Structures

Midterm 1 Exam – Solutions

February 28, 2023

1. The cardinality of a set A is its size and is denoted by $|A|$.

   (a) Let A and B be two sets. Assume that

   $$|A \setminus B| = |B \setminus A|$$

   What is the relationship between the cardinalities of A and B?

   Answer: The cardinality of A is the same as the cardinality of B. That is, $|A| = |B|$.

   Proof: Let $x = |A \setminus B| = |B \setminus A|$ be the cardinality of $A \setminus B$ and $B \setminus A$ and let $y = |A \cap B|$ be the cardinality of the intersection of the sets A and B.
   Observe that the two sets $A \setminus B$ and $A \cap B$ are disjoint. As a result, $|A| = |A \setminus B| + |A \cap B|$ and $|B| = |B \setminus A| + |A \cap B|$. It follows that $|A| = |B| = x + y$.

   (b) Let S and T be two sets. Suppose that the cardinality of the union of the two sets S and T is larger by exactly one than the cardinality of the intersection of S and T:

   $$|S \cup T| = |S \cap T| + 1$$

   What is the relationship between the sets S and T?

   Answer: Either $S \subset T$ and $|T| = |S| + 1$ or $T \subset S$ and $|S| = |T| + 1$.

   Proof: Let $y = |S \cap T|$ be the cardinality of the intersection of the sets S and T, let $x_s = |S \setminus T|$ be the cardinality of the set $S \setminus T$, and let $x_t = |T \setminus S|$ be the cardinality of the set $T \setminus S$. It follows that $y + x_s + x_t = |S \cup T|$ is the cardinality of the union of the sets S and T.
   Since $|S \cup T| = |S \cap T| + 1$, it follows that $y + x_s + x_t = y + 1$ which implies that $x_s + x_t = 1$. But both x_s and x_t are nonnegative integers implying that one of them must equal 0 and one of them must equal 1.
   If $x_t = 1$ and $x_s = 0$ then $S \subset T$ and $|T| = |S|+1$ and if $x_s = 1$ and $x_t = 0$ then $T \subset S$ and $|S| = |T|+1$.

2. For a positive integer k, let $A_k = \{-k, -(k-1), \ldots, -1, 0, 1, \ldots, (k-1), k\}$ be the set of all integers between $-k$ and k.

   (a) Find the explicit form of each one of the sets A_1, A_2, and A_3.

   Answer: $A_1 = \{-1, 0, 1\}$, $A_2 = \{-2, -1, 0, 1, 2\}$, and $A_3 = \{-3, -2, -1, 0, 1, 2, 3\}$

   (b) Find the explicit form of the set $S_3 = A_1 \cap A_2 \cap A_3$.

   Answer: $S_3 = \{-1, 0, 1\} = A_1$ because $A_1 \subset A_2 \subset A_3$.

   (c) Find the explicit form of the set $T_3 = A_1 \cup A_2 \cup A_3$.

   Answer: $T_3 = \{-3, -2, -1, 0, 1, 2, 3\} = A_3$ because $A_1 \subset A_2 \subset A_3$.

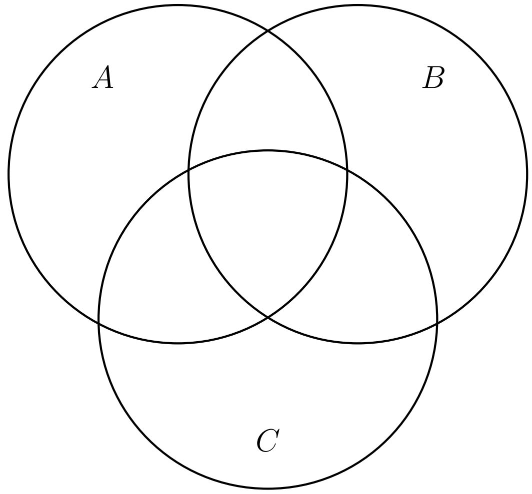
   (d) For a positive integer n, simplify $S_n = A_1 \cap A_2 \cap \cdots \cap A_n$.

   Answer: $S_n = \{-1, 0, 1\} = A_1$ because $A_1 \subset A_2 \subset \cdots \subset A_n$.

   (e) For a positive integer n, simplify $T_n = A_1 \cup A_2 \cup \cdots \cup A_n$.

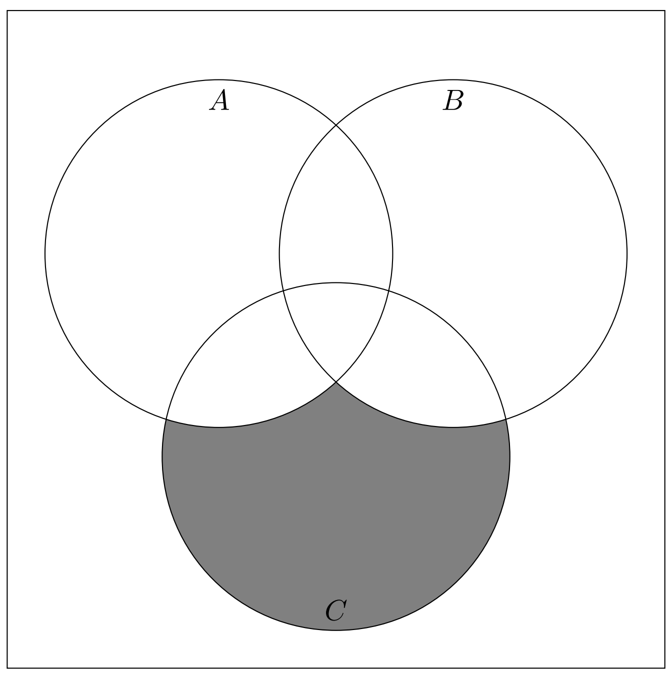
   Answer: $T_n = \{-n, -(n-1), \ldots, -1, 0, 1, \ldots, (n-1), n\} = A_n$ because $A_1 \subset A_2 \subset \cdots \subset A_n$.

3. Let A, B, and C be three non-empty sets such that $\mathbf{C} \subseteq (\mathbf{A} \cup \mathbf{B})$.



(a) In the above Venn-diagram mark the zone that MUST be empty given that $C \subseteq (A \cup B)$.

Answer: The zone $C \setminus (A \cup B) = \overline{A}\overline{B}C$ must be empty because no object may belong to C if it does not belong to either A or B.



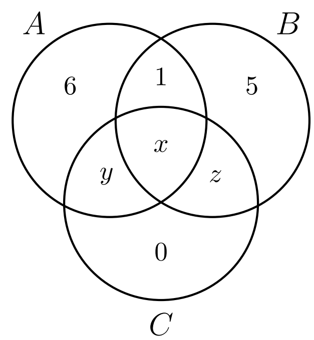
(b) Given that $C \subseteq (A \cup B)$, the following additional data is known:

- A contains 14 objects ($|A| = 14$), B contains 12 objects ($|B| = 12$), and C contains 9 objects ($|C| = 9$).
- There are 6 objects that belong only to A ($|A \setminus (B \cup C)| = 6$), there are 5 objects that belong only to B ($|B \setminus (A \cup C)| = 5$), and there is only one object in the intersection of A and B that is not in C ($|(A \cap B) \setminus C| = 1$).

i. What is the size of the intersection of all three sets ($A \cap B \cap C$)?

ii. What is the size of the union of all three sets ($A \cup B \cup C$)?

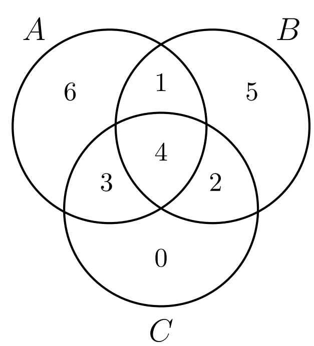
Answer: The diagram below reflects the data given in the second bullet and the answer from part (a). Denote by $x = |A \cap B \cap C|$, by $y = |(A \cap C) \setminus B|$, and by $z = |(B \cap C) \setminus A|$ the sizes of the zones for which the data was not explicit about their sizes.



The given size of A implies that $14 = 6 + 1 + x + y$, the given size of B implies that $12 = 5 + 1 + x + z$, and the given size of C implies that $9 = 0 + x + y + z$. All together, there are three equations with three variables

$$x + y = 7 \quad (1)$$
$$x + z = 6 \quad (2)$$
$$x + y + z = 9 \quad (3)$$

Equations (1) and (3) imply that $z = 2$ while equations (2) and (3) imply that $y = 3$. These values of y and z and equation (3) imply that $x = 4$. See the diagram below for the sizes of all the seven zones in the union of all three sets.



Final answers to parts (i) and (ii):

i. The size of the intersection of all three sets is $\mathbf{4} = |A \cap B \cap C|$.

ii. The size of the union of all three sets is $\mathbf{21} = 6 + 5 + 4 + 3 + 2 + 1 = |A \cup B \cup C|$.

4. Let x, y, and z be three boolean variables.

(a) Define a boolean formula that must contain x, y, and z that is satisfied by **exactly one** TRUE/FALSE assignment to x, y, and z.

Answer: The formula

$$x \wedge y \wedge z$$

is satisfied only by **one** assignment that assigns TRUE to all three variables.

(b) Define a boolean formula that must contain x, y, and z that is satisfied by **exactly seven** TRUE/FALSE assignments to x, y, and z.

Answer: The formula

$$x \vee y \vee z$$

is not satisfied only by **one** assignment that assigns FALSE to all three variables. Therefore, the other **seven** assignments satisfy this formula.

(c) Define a boolean formula that must contain x, y, and z for which **exactly four** TRUE/FALSE assignments to x, y, and z satisfy the formula.

Answer: The formula

$$x \wedge (y \vee \neg y) \wedge (z \vee \neg z)$$

is satisfies if and only if x is assigned TRUE which happens in four out of the eight possible assignments. This is because both $(y \vee \neg y)$ and $(z \vee \neg z)$ are tautologies (always TRUE).
Same thing with the formula

$$x \vee (y \wedge \neg y) \vee (z \wedge \neg z)$$

Here it is because both $(y \wedge \neg y)$ and $(z \wedge \neg z)$ are contradictions (always FALSE).
There are many other such formulas For example $x \equiv y \equiv z$ and $x \oplus y \oplus z$.

5. The following is the truth table of the function $\mathcal{EQUIV}$ (denoted by $\equiv$) with the boolean variables x and y.

x	y	$x \equiv y$
T	T	T
T	F	F
F	T	F
F	F	T

(a) Let x, y, and z be three boolean variables. Prove that the following identity is correct:

$$((x \equiv y) \equiv z) \;=\; (y \equiv (x \equiv z))$$

Proof: The left table below is the truth table of the left side of the identity while the right table below is the truth table of the right side of the identity. The identity is correct because the last columns of both tables are identical.

x	y	z	$x \equiv y$	$(x \equiv y) \equiv z$
T	T	T	T	T
T	T	F	T	F
T	F	T	F	F
T	F	F	F	T
F	T	T	F	F
F	T	F	F	T
F	F	T	T	T
F	F	F	T	F

x	y	z	$x \equiv z$	$y \equiv (x \equiv z)$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	F
T	F	F	F	T
F	T	T	F	F
F	T	F	T	T
F	F	T	F	T
F	F	F	T	F

Remark: The function $\mathcal{EQUIV}$ is commutative and associative. Therefore, for any integer n, any order for evaluating the formula $(x_1 \equiv x_2 \equiv \cdots \equiv x_n)$ with the n variables $x_1, x_2, \ldots, x_n$ will yield the same result.

The truth tables above demonstrates this for two out of the three possible orders to evaluate the formula $(x \equiv y \equiv z)$. In the left side of the identity the first pair of variables to be evaluated is (x, y) and in the right side of the identity the first pair of variables to be evaluated is (x, z). The third possible way is when the pair of variables (y, z) is evaluated first as is the case in the formulas $((y \equiv z) \equiv x)$ and $(x \equiv (y \equiv z))$.

(b) Let $x_1, x_2, \ldots, x_n$ be n ($n \geq 2$) boolean variables. Consider the following boolean formula:

$$P = (x_1 \equiv x_2 \equiv \cdots \equiv x_n)$$

Notations: For $1 \leq i \leq n$, let $P_i = (x_1 \equiv x_2 \equiv \cdots \equiv x_i)$. In particular, $P_1 = x_1$ and $P_n = P$.

Evaluating P: It is done from left to right. As a result, the evaluation process is as follows

$$\begin{array}{rcl} P_1 &=& (x_1) &=& (x_1) \\ P_2 &=& (x_1 \equiv x_2) &=& (P_1 \equiv x_2) \\ P_3 &=& (x_1 \equiv x_2 \equiv x_3) &=& (P_2 \equiv x_3) \\ & \vdots & & & \\ P_i &=& (x_1 \equiv x_2 \equiv \cdots \equiv x_i) &=& (P_{i-1} \equiv x_i) \\ & \vdots & & & \\ P = P_n &=& (x_1 \equiv x_2 \equiv \cdots \equiv x_{n-1} \equiv x_n) &=& (P_{n-1} \equiv x_n) \end{array}$$

i. What is the value of P (T or F) when all the n variables are TRUE?

Answer: P is TRUE.

Proof: Since $x_1 = T$, it follows by definition that $P_1 = T$. Next, $P_2 = (P_1 \equiv x_2) = T$ because $(T \equiv T) = T$. Next, $P_3 = (P_2 \equiv x_3) = T$ because $(T \equiv T) = T$. Continue this way to show that $P_4 = P_5 = \cdots = P_n = T$.
The answer follows because in particular $P = P_n$ is TRUE.

ii. What is the value of P (T or F) when all the n variables are FALSE?

Answer: P is FALSE when n is odd and P is TRUE when n is even.

Proof sketch: Since $x_1 = F$, it follows that $P_1 = F$. Next, $P_2 = (P_1 \equiv x_2) = T$ because $(F \equiv F) = T$. Next, $P_3 = (P_2 \equiv x_3) = F$ because $(T \equiv F) = F$.
In general, the value of P_i alternates from F to T and then back to F depending on the parity of i. If i is odd, then $P_i = F$ and if i is even then $P_i = T$.
As a result, $P = P_n$ is FALSE for an odd n while $P = P_n$ is TRUE for an even n.

iii. What is the value of P (TRUE or FALSE) when out of the n variables exactly k ($0 \leq k \leq n$) are FALSE?

Answer: P is FALSE when k is odd and P is TRUE when k is even.

Proof sketch: Observe that $P_i = (P_{i-1} \equiv x_i)$ is different than P_{i-1} only if $x_i = F$. This is because $(R \equiv T) = R$ while $(R \equiv F) = \neg R$ for either $R = T$ or $R = F$.
As a result, the value of $(x_1 \equiv x_2 \equiv \cdots \equiv x_n)$ is the same as the value of $(x_{i_1} \equiv x_{i_2} \equiv \cdots \equiv x_{i_k})$ where $1 \leq i_1, i_2, \ldots, i_k \leq n$ are all the indices i for which $x_i = F$. This is because by the observation, all the other x_i (those for which $x_i = T$) do not change the value of P_{i-1}.
The answer is a corollary of the result from part (ii).

6. Only one of Alice, Bob or Charlie failed the test while the other two got an A on the test.
   - Alice says that Bob failed.
   - Bob says that Alice is lying.
   - Charlie says that he did not fail.

   Only one of them is telling the truth.

   Who failed the test and who is the one telling the truth?

   Answer: Charlie failed the test and Bob is the only one telling the truth.

   Proof I: Alice's claim ("Bob failed") and Bob's claim ("Alice is lying") are mutually exclusive; one must be true and the other must be false. This means that the single truth-teller required by the rules must be either Alice or Bob. This forces the third person, Charlie, to be lying. Since Charlie claims he did not fail, his lie confirms the opposite: Charlie is the one who failed the test.

   Knowing that Charlie failed means Bob must have passed with an A. This reveals that Alice's statement ("Bob failed") is false. Consequently, Alice is a liar. Since Alice is lying, Bob's statement ("Alice is lying") must be true, making Bob the sole truth-teller.

   Proof II: Assume Charlie is telling the truth. According to the rules, this would make him the only truth-teller, meaning both Alice and Bob must be lying. However, a contradiction arises when examining Bob's statement. If Alice is lying (as the assumption requires), then Bob's statement, "Alice is lying" would be factually true. This scenario results in two truth-tellers (Charlie and Bob), which violates the rule that there can be only one. Therefore, the initial assumption is impossible. Charlie must be lying.

   Since Charlie is lying about his claim that he "did not fail," it means Charlie is the one who failed the test. This confirms that both Alice and Bob passed. As a result, Alice's statement ("Bob failed") is false, and Bob's statement ("Alice is lying") is true, making Bob the single truth-teller.