CISC 2210 – Introduction to Discrete Structures

Makeup Midterm Exam 2 – Solutions

April 25, 2023

1. Prove the correctness of the following identity for any $n \geq 1$.

$$\sum_{i=1}^{n} i + \sum_{i=1}^{n} 2i + \sum_{i=1}^{n} 3i \;=\; 3n(n+1)$$

Proof by induction:

- *Notations.*

$$\begin{aligned} L(n) &= \sum_{i=1}^{n} i + \sum_{i=1}^{n} 2i + \sum_{i=1}^{n} 3i \\ R(n) &= 3n(n+1) \end{aligned}$$

- *Induction base.* Prove that $L(1) = R(1)$:

$$L(1) = 1 + 2 \cdot 1 + 3 \cdot 1 = 1 + 2 + 3 = 6 = 3 \cdot 2 = 3(1 \cdot 2) = R(1)$$

- *Induction hypothesis.* Assume that $L(k) = R(k)$ for $k \geq 1$:

$$\sum_{i=1}^{n} i + \sum_{i=1}^{n} 2i + \sum_{i=1}^{n} 3i \;=\; 3k(k+1)$$

- *Inductive step.* Prove that $L(k+1) = R(k+1)$ for $k \geq 1$:

$$\begin{aligned} L(k+1) &= \sum_{i=1}^{k+1} i + \sum_{i=1}^{k+1} 2i + \sum_{i=1}^{k+1} 3i \\ &= \left(\sum_{i=1}^{k} i + (k+1)\right) + \left(\sum_{i=1}^{k} 2i + 2(k+1)\right) + \left(\sum_{i=1}^{k} 3i + 3(k+1)\right) \\ &= \left(\sum_{i=1}^{k} i + \sum_{i=1}^{k} 2i + \sum_{i=1}^{k} 3i\right) + ((k+1) + 2(k+1) + 3(k+1)) \\ &= L(k) + ((k+1) + 2(k+1) + 3(k+1)) \\ &= R(k) + ((k+1) + (2k+2) + (3k+3)) \\ &= 3k(k+1) + (6k+6) \\ &= 3(k^2+k) + 3(2k+2) \\ &= 3(k^2+k+2k+2) \\ &= 3(k^2+3k+2) \\ &= 3(k+1)(k+2) \\ &= R(k+1) \end{aligned}$$

A proof without induction:

$$\begin{aligned} \sum_{i=1}^{n} i + \sum_{i=1}^{n} 2i + \sum_{i=1}^{n} 3i &= \sum_{i=1}^{n} (i + 2i + 3i) \\ &= \sum_{i=1}^{n} (6i) \\ &= 6 \sum_{i=1}^{n} i \\ &= 6 \cdot \frac{n(n+1)}{2} \\ &= 3n(n+1) \end{aligned}$$

2. Consider the following recurrence for integers $n \geq 1$:

$$M(n) = \begin{cases} 3 & \text{for } n = 1 \\ 2M(n-1) + 3 & \text{for } n \geq 2 \end{cases}$$

Prove that for $n \geq 1$

$$M(n) = 3(2^n - 1)$$

Proof by induction:

- *Induction base.* For $n = 1$

$$M(1) = 3(2^1 - 1) = 3(2 - 1) = 3 \cdot 1 = 3$$

- *Induction hypothesis.* Assume that for $n > 1$

$$M(n-1) = 3(2^{n-1} - 1)$$

- *Inductive step.* Prove that for $n > 1$

$$M(n) = 3(2^n - 1)$$

$$\begin{aligned} M(n) &= 2M(n-1) + 3 \\ &= 2(3(2^{n-1} - 1)) + 3 \\ &= 3(2(2^{n-1} - 1)) + 3 \\ &= 3(2(2^{n-1} - 1) + 1) \\ &= 3(2^n - 2 + 1) \\ &= 3(2^n - 1) \end{aligned}$$

3. An ice cream shop sells several flavors of ice cream. You plan to buy three scoops in a cone and you **care about their order**. You might or might not buy a particular flavor more than once.

(a) There are only 4 flavors.

i. Find the number of 3-scoop cones you can buy.

Answer: There are 4 options for each one of the 3 scoops. Therefore, the total number of 3-scoop cones is

$$4 \times 4 \times 4 = \mathbf{64}$$

ii. Find the number of 3-scoop cones you can buy in which the bottom scoop must be chocolate.

Answer: There are 4 options for each one of the top 2 scoops. Therefore, the total number of 3-scoop cones in which the bottom scoop must be chocolate is

$$4 \times 4 = \mathbf{16}$$

iii. Find the number of 3-scoop cones you can buy in which the bottom scoop cannot be chocolate.

Answer: There are 4 options for each one of the top 2 scoops and only 3 options for the bottom scoop. Therefore, the total number of 3-scoop cones in which the bottom scoop cannot be chocolate is

$$4 \times 4 \times 3 = \mathbf{48}$$

iv. How do the three questions above relate to each other?

Answer: Every option from part (i) is counted either in part (ii) when the bottom scoop must be chocolate or in part (iii) when the bottom scoop cannot be chocolate, but not in both parts together. Indeed,

$$\mathbf{64} = 4 \times 4 \times 4 = 4 \times 4 \times (1 + 3) = 4 \times 4 \times 1 + 4 \times 4 \times 3 = \mathbf{16} + \mathbf{48}$$

(b) There are n flavors for an integer $n \geq 3$.

i. Find the number of 3-scoop cones you can buy.

Answer: There are n options for each one of the 3 scoops. Therefore, the total number of 3-scoop cones is

$$n \times n \times n = \mathbf{n^3}$$

ii. Find the number of 3-scoop cones you can buy in which the bottom scoop must be chocolate.

Answer: There are n options for each one of the top 2 scoops. Therefore, the total number of 3-scoop cones in which the bottom scoop must be chocolate is

$$n \times n = \mathbf{n^2}$$

iii. Find the number of 3-scoop cones you can buy in which the bottom scoop cannot be chocolate.

Answer: There are n options for each one of the top 2 scoops and only $n - 1$ options for the bottom scoop. Therefore, the total number of 3-scoop cones in which the bottom scoop cannot be chocolate is

$$n \times n \times (n - 1) = \mathbf{n^3} - \mathbf{n^2}$$

iv. How do the three questions above relate to each other?

Answer: Every option from part (i) is counted either in part (ii) when the bottom scoop must be chocolate or in part (iii) when the bottom scoop cannot be chocolate, but not in both parts together. Indeed,

$$\mathbf{n^3} = n \times n \times n = n \times n \times (1 + (n - 1)) = n \times n \times 1 + n \times n \times (n - 1) = \mathbf{n^2} + (\mathbf{n^3} - \mathbf{n^2})$$

4. Simplify the following expression into an expression that does not contain binomial coefficients, factorials, and fractions.

$$\binom{n+1}{2} - \binom{n-1}{2} + 1$$

Answer:

$$\binom{n+1}{2} - \binom{n-1}{2} + 1 = \mathbf{2n}$$

Proof 1: The formula for $\binom{n}{2}$ implies that $\binom{n+1}{2} = \frac{(n+1)n}{2}$ and that $\binom{n-1}{2} = \frac{(n-1)(n-2)}{2}$. Therefore

$$\begin{aligned}
\binom{n+1}{2} - \binom{n-1}{2} + 1 &= \frac{(n+1)n}{2} - \frac{(n-1)(n-2)}{2} + 1 \\
&= \frac{n^2+n}{2} - \frac{n^2-3n+2}{2} + \frac{2}{2} \\
&= \frac{n^2+n-n^2+3n-2+2}{2} \\
&= \frac{4n}{2} \\
&= \mathbf{2n}
\end{aligned}$$

Proof 2: The recursive formula for binomial coefficients implies that $\binom{n+1}{2} = \binom{n}{1} + \binom{n}{2}$ and that $\binom{n-1}{2} = \binom{n}{2} - \binom{n-1}{1}$. Therefore

$$\begin{aligned}
\binom{n+1}{2} - \binom{n-1}{2} + 1 &= \left(\binom{n}{1} + \binom{n}{2}\right) - \left(\binom{n}{2} - \binom{n-1}{1}\right) + 1 \\
&= \binom{n}{1} + \binom{n}{2} - \binom{n}{2} + \binom{n-1}{1} + 1 \\
&= \binom{n}{1} + \binom{n-1}{1} + 1 \\
&= n + (n-1) + 1 \\
&= \mathbf{2n}
\end{aligned}$$

5. A bag contains 6 marbles: 1 Blue (B) marble, 2 Red (R) marbles, and 3 Green (G) marbles. Two random marbles are drawn from the bag.

(a) What is the probability that both marbles are of the same color?

Answer: $\mathbf{\frac{4}{15}}$

Counting proof: Denote the marbles by B_1, R_1, R_2, G_1, G_2, and G_3. There are $\binom{6}{3} = 15$ equal probability pairs of two marbles. The pairs with the same color marbles are emphasized.

$$(B_1R_1) \;\; (B_1R_2) \;\; (B_1G_1) \;\; (B_1G_2) \;\; (B_1G_3)$$
$$(\mathbf{R_1R_2}) \;\; (R_1G_1) \;\; (R_1G_2) \;\; (R_1G_3) \;\; (R_2G_1)$$
$$(R_2G_2) \;\; (R_2G_3) \;\; (\mathbf{G_1G_2}) \;\; (\mathbf{G_1G_3}) \;\; (\mathbf{G_2G_3})$$

Out of these 15 pairs, 4 pairs are of the same color while 11 pairs are of different colors. Therefore, the probability that both marbles are of the same color is **4/15**.

Probability proof: There are two disjoint cases: (i) with probability $\binom{2}{2}/\binom{6}{2}$ the two marbles are Red and (ii) with probability $\binom{3}{2}/\binom{6}{2}$ the two marbles are Green. Since there is only one Blue marble, it cannot be the case that both marbles are Blue. Therefore, the probability that both marbles are of the same color is

$$\frac{\binom{2}{2}}{\binom{6}{2}} + \frac{\binom{3}{2}}{\binom{6}{2}} = \frac{1}{15} + \frac{3}{15} = \mathbf{\frac{4}{15}}$$

(b) What is the probability that both marbles are of different colors?

Answer: $\mathbf{\frac{11}{15}}$

Counting proof: Denote the marbles by B_1, R_1, R_2, G_1, G_2, and G_3. There are $\binom{6}{3} = 15$ equal probability pairs of two marbles. The pairs with different colors are emphasized.

$$(\mathbf{B_1R_1}) \;\; (\mathbf{B_1R_2}) \;\; (\mathbf{B_1G_1}) \;\; (\mathbf{B_1G_2}) \;\; (\mathbf{B_1G_3})$$
$$(R_1R_2) \;\; (\mathbf{R_1G_1}) \;\; (\mathbf{R_1G_2}) \;\; (\mathbf{R_1G_3}) \;\; (\mathbf{R_2G_1})$$
$$(\mathbf{R_2G_2}) \;\; (\mathbf{R_2G_3}) \;\; (G_1G_2) \;\; (G_1G_3) \;\; (G_2G_3)$$

Out of these 15 pairs, 11 pairs are of different colors while 4 pairs are of the same color. Therefore, the probability that both marbles are of different colors is **11/15**.

Probability proof: There are three disjoint cases: (i) with probability $(1 \cdot 2)/\binom{6}{2}$ the two marbles are Blue and Red; (ii) with probability $(1 \cdot 3)/\binom{6}{2}$ the two marbles are Blue and Green; and (iii) with probability $(2 \cdot 3)/\binom{6}{2}$ the two marbles are Red and Green. Therefore, the probability that both marbles are of different colors is

$$\frac{1 \cdot 2}{\binom{6}{2}} + \frac{1 \cdot 3}{\binom{6}{2}} + \frac{2 \cdot 3}{\binom{6}{2}} = \frac{2}{15} + \frac{3}{15} + \frac{6}{15} = \mathbf{\frac{11}{15}}$$

Another probability proof: Let S be the event that both marbles have the same color and let D be the even that both marbles are of different colors. It follows that $D = \overline{S}$. Part (a) proved that $p(S) = 4/15$ and therefore

$$p(D) = 1 - p(S) = 1 - \frac{4}{15} = \mathbf{\frac{11}{15}}$$

(c) What is the probability that both marbles are of the same color given that at least one of them is Green?

Answer: $\mathbf{\frac{1}{4}}$

Counting proof: Denote the marbles by B_1, R_1, R_2, G_1, G_2, and G_3. There are $\binom{3}{2} = 3$ pairs of marbles out of the $\binom{6}{2} = 15$ equal probability pairs of marbles in which none of the marbles is Green. This implies that there are $12 = 15 - 3$ equal probability pairs of marbles in which at least one of the marbles is Green. The pairs with the same color are emphasized.

$$(B_1G_1)\ \ (B_1G_2)\ \ (B_1G_3)\ \ (R_1G_1)\ \ (R_1G_2)\ \ (R_1G_3)$$
$$(R_2G_1)\ \ (R_2G_2)\ \ (R_2G_3)\ \ (\mathbf{G_1G_2})\ \ (\mathbf{G_1G_3})\ \ (\mathbf{G_2G_3})$$

Out of these 12 pairs, 3 pairs are of the same color while 9 pairs are of different colors. Therefore, the probability that both marbles are of the same color given that at least one of them is Green is $3/12 = \mathbf{1/4}$.

Probability proof: Let A be the event that at least one of the marbles is Green. The counting proof showed that $P(A) = 12/15 = 4/5$. Recall that S is the event that both marbles are of the same color. Since there are $\binom{6}{2} = 15$ equal probability pairs of marbles and in $\binom{3}{2}$ of them the two marbles are Green, it follows that $p(A \cap S) = 3/15 = 1/5$. By definition, $p(S|A)$ denotes the probability that both marbles are of the same color given that at least one of them is Green. Bayes' Theorem implies that

$$p(S|A) = \frac{p(A \cap S)}{p(A)} = \frac{1/5}{4/5} = \frac{5}{20} = \mathbf{\frac{1}{4}}$$

(d) What is the probability that both marbles are of different colors given that at least one of them is Red?

Answer: $\mathbf{\frac{8}{9}}$

Counting proof: Denote the marbles by B_1, R_1, R_2, G_1, G_2, and G_3. There are $\binom{4}{2} = 6$ pairs of marbles out of the $\binom{6}{2} = 15$ equal probability pairs of marbles in which none of the marbles is Red. This implies that there are $9 = 15 - 6$ equal probability pairs of marbles in which at least one of the marbles is Red. The pairs with different colors are emphasized.

$$(\mathbf{B_1R_1})\ \ (\mathbf{B_1R_2})\ \ (R_1R_2)\ \ (\mathbf{R_1G_1})\ \ (\mathbf{R_1G_2})\ \ (\mathbf{R_1G_3})\ \ (\mathbf{R_2G_1})\ \ (\mathbf{R_2G_2})\ \ (\mathbf{R_2G_3})$$

Out of these 9 pairs, 8 pairs are of different colors while only one pair is of the same color. Therefore, the probability that both marbles are of different colors given that at least one of them is Red is $\mathbf{8/9}$.

Probability proof: Let C be the event that at least one of the marbles is Red. The counting proof showed that $P(C) = 9/15 = 3/5$. Recall that D is the event that both marbles are of different colors. Since there are $\binom{6}{2} = 15$ equal probability pairs of marbles and in $8 = 2 \cdot 4$ of them exactly one marble is Red, it follows that $p(C \cap D) = 8/15$. By definition, $p(D|C)$ denotes the probability that both marbles are of different colors given that at least one of them is Red. Bayes' Theorem implies that

$$p(D|C) = \frac{p(C \cap D)}{p(C)} = \frac{8/15}{3/5} = \frac{40}{45} = \mathbf{\frac{8}{9}}$$