CISC 2210 TR11 – Introduction to Discrete Structures

Midterm Exam 2 – Solutions

April 18, 2023

1. Prove the correctness of the following identity for any $n \geq 1$.

$$\sum_{i=1}^{n}(3i+1) - \sum_{i=1}^{n}(i+1) \;=\; n(n+1)$$

Proof by induction:

- *Notations.*

$$\begin{aligned} L(n) &= \sum_{i=1}^{n}(3i+1) - \sum_{i=1}^{n}(i+1) \\ R(n) &= n(n+1) \end{aligned}$$

- *Induction base.* Prove that $L(1) = R(1)$:

$$L(1) = (3 \cdot 1 + 1) - (1 + 1) = 4 - 2 = 2 = 1 \cdot 2 = R(1)$$

- *Induction hypothesis.* Assume that $L(k) = R(k)$ for $k \geq 1$:

$$\sum_{i=1}^{k}(3i+1) - \sum_{i=1}^{k}(i+1) \;=\; k(k+1)$$

- *Inductive step.* Prove that $L(k+1) = R(k+1)$ for $k \geq 1$:

$$\begin{aligned} L(k+1) &= \sum_{i=1}^{k+1}(3i+1) - \sum_{i=1}^{k+1}(i+1) \\ &= \left(\sum_{i=1}^{k}(3i+1) + (3(k+1)+1)\right) - \left(\sum_{i=1}^{k}(i+1) + ((k+1)+1)\right) \\ &= \left(\sum_{i=1}^{k}(3i+1) - \sum_{i=1}^{k}(i+1)\right) + ((3(k+1)+1) - ((k+1)+1)) \\ &= L(k) + (3k+3+1-k-1-1) \\ &= R(k) + (2k+2) \\ &= k(k+1) + (2k+2) \\ &= k^2 + k + 2k + 2 \\ &= k^2 + 3k + 2 \\ &= (k+1)(k+2) \\ &= R(k+1) \end{aligned}$$

A proof without induction:

$$\begin{aligned} \sum_{i=1}^{n}(3i+1) - \sum_{i=1}^{n}(i+1) &= \sum_{i=1}^{n}((3i+1) - (i+1)) \\ &= \sum_{i=1}^{n}(2i) \\ &= 2\sum_{i=1}^{n} i \\ &= 2 \cdot \frac{n(n+1)}{2} \\ &= n(n+1) \end{aligned}$$

2. Consider the following recurrence for integers $n \geq 1$:

$$M(n) = \begin{cases} 1 & \text{for } n = 1 \\ 3M(n-1) + 1 & \text{for } n \geq 2 \end{cases}$$

Prove that for $n \geq 1$

$$M(n) = \frac{3^n - 1}{2}$$

Proof by induction:

- *Induction base.* For $n = 1$

$$M(1) = \frac{3^1 - 1}{2} = \frac{3 - 1}{2} = \frac{2}{2} = 1$$

- *Induction hypothesis.* Assume that for $n > 1$

$$M(n-1) = \frac{3^{n-1} - 1}{2}$$

- *Inductive step.* Prove that for $n > 1$

$$M(n) = \frac{3^n - 1}{2}$$

$$\begin{aligned} M(n) &= 3M(n-1) + 1 \\ &= 3 \cdot \frac{3^{n-1} - 1}{2} + 1 \\ &= \frac{3(3^{n-1} - 1)}{2} + \frac{2}{2} \\ &= \frac{3^n - 3 + 2}{2} \\ &= \frac{3^n - 1}{2} \end{aligned}$$

3. You are ordering a pizza. You plan to add toppings (order does not matter) chosen from a list of toppings that contains pineapple as one of the options.

   (a) You want 3 different toppings chosen from a list of 5 toppings.

      i. How many topping combinations do you have for your pizza?

         Answer: The 3 toppings must be different and their order does not matter. Therefore, the number of topping combinations is

$$\binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{5!}{3!2!} = \frac{120}{6 \cdot 2} = \frac{120}{12} = \mathbf{10}$$

      ii. How many topping combinations do you have for your pizza if you would never consider pineapple as one of your toppings?

         Answer: Since pineapple is not an option, it follows that there are only 4 available toppings instead of 5. Therefore, the number of topping combinations in which pineapple is not an option is

$$\binom{4}{3} = \frac{4!}{3!(4-3)!} = \frac{4!}{3!1!} = \frac{24}{6 \cdot 1} = \frac{24}{6} = \mathbf{4}$$

      iii. How many topping combinations do you have for your pizza if you insist on having pineapple as one of your toppings?

         Answer: Since pineapple must be one of the selected toppings, it follows that the other 2 toppings must be selected from the remaining 4 toppings. Therefore, the number of topping combinations in which pineapple must be one of them is

$$\binom{4}{2} = \frac{4!}{2!(4-2)!} = \frac{4!}{2!2!} = \frac{24}{2 \cdot 2} = \frac{24}{4} = \mathbf{6}$$

      iv. How do the three questions above relate to each other?

         Answer: Pineapple is either one of the toppings or not. Therefore, the answer to part (i) must be equal to the sum of the answers in part (ii) and part (iii). Indeed,

$$\mathbf{10} = \binom{5}{3} = \binom{4}{3} + \binom{4}{2} = \mathbf{4} + \mathbf{6}$$

(b) You want k **different** toppings chosen from a list of n toppings for some integers $n > k \geq 1$ in which one of the toppings is pineapple.

i. How many topping combinations do you have for your pizza?

Answer: The k toppings must be different and their order does not matter. Therefore, the number of topping combinations is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

ii. How many topping combinations do you have for your pizza if you would never consider pineapple as one of your toppings?

Answer: Since pineapple is not an option, it follows that there are only $n-1$ available toppings instead of n. Therefore, the number of topping combinations in which pineapple is not an option is

$$\binom{n-1}{k} = \frac{(n-1)!}{k!(n-k-1)!}$$

iii. How many topping combinations do you have for your pizza if you insist on having pineapple as one of your toppings?

Answer: Since pineapple must be one of the selected toppings, it follows that the other $k-1$ toppings must be selected from the remaining $n-1$ toppings. Therefore, the number of topping combinations in which pineapple must be one of them is

$$\binom{n-1}{k-1} = \frac{(n-1)!}{(k-1)!(n-k)!}$$

iv. How do the three questions above relate to each other?

Answer: Pineapple is either one of the toppings or not. Therefore, the answer to part (i) must be equal to the sum of the answers in part (ii) and part (iii). Indeed, this is the fundamental recursive formula for the binomial coefficient $\binom{n}{k}$:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

4. Simplify the following expression into an expression that does not contain binomial coefficients, factorials, and fractions.

$$\binom{n}{2} + \binom{n-1}{2}$$

Answer:

$$\binom{n}{2} + \binom{n-1}{2} = (\mathbf{n-1})^{\mathbf{2}}$$

Proof 1: The formula for $\binom{n}{2}$ implies that $\binom{n}{2} = \frac{n(n-1)}{2}$ and that $\binom{n-1}{2} = \frac{(n-1)(n-2)}{2}$. Therefore

$$\begin{aligned}
\binom{n}{2} + \binom{n-1}{2} &= \frac{n(n-1)}{2} + \frac{(n-1)(n-2)}{2} \\
&= \frac{n(n-1) + (n-1)(n-2))}{2} \\
&= \frac{(n-1)(n + (n-2))}{2} \\
&= \frac{(n-1)(2n-2)}{2} \\
&= \frac{2(n-1)(n-1)}{2} \\
&= (n-1)(n-1) \\
&= (\mathbf{n-1})^{\mathbf{2}}
\end{aligned}$$

Proof 2: The recursive formula for binomial coefficients implies that $\binom{n}{2} = \binom{n-1}{1} + \binom{n-1}{2}$. Therefore

$$\begin{aligned}
\binom{n}{2} + \binom{n-1}{2} &= \left(\binom{n-1}{1} + \binom{n-1}{2}\right) + \binom{n-1}{2} \\
&= \binom{n-1}{1} + \left(\binom{n-1}{2} + \binom{n-1}{2}\right) \\
&= (n-1) + 2\frac{(n-1)(n-2))}{2} \\
&= (n-1) + (n-1)(n-2) \\
&= (n-1)(1 + (n-2)) \\
&= (n-1)(n-1) \\
&= (\mathbf{n-1})^{\mathbf{2}}
\end{aligned}$$

5. Two fair dice are thrown: one is a 5-sided dice labeled with the numbers $1, 2, 3, 4, 5$ on its 5 faces and one is a 4-sided dice labeled with the numbers $1, 2, 3, 4$ on its 4 faces.

(a) What is the probability that the sum of the two shown numbers is even?

Answer: $\mathbf{\frac{1}{2}}$

Counting proof: There are 20 equal probability combinations for the two shown numbers. The following table shows the 20 possible sums. The even sums are emphasized.

Sum	1	2	3	4	5
1	**2**	3	**4**	5	**6**
2	3	**4**	5	**6**	7
3	**4**	5	**6**	7	**8**
4	5	**6**	7	**8**	9

Out of these 20 sums, 10 are even and 10 are odd. Therefore, the probability that the sum of the two shown numbers is even is

$$\frac{10}{20} = \mathbf{\frac{1}{2}}$$

Probability proof: Let A be the event that the sum is even. This happens if and only if either both shown numbers are even or both shown numbers are odd. The probability that the 5-sided dice shows an even number is $2/5$ and the probability that it shows an odd number is $3/5$ while the probability that the 4-sided dice shows an even number is $1/2$ and the probability that it shows an odd number is $1/2$. Therefore,

$$p(A) = \frac{2}{5} \cdot \frac{1}{2} + \frac{3}{5} \cdot \frac{1}{2} = \frac{2}{10} + \frac{3}{10} = \frac{5}{10} = \mathbf{\frac{1}{2}}$$

(b) What is the probability that the product of the two shown numbers is even?

Answer: $\mathbf{\frac{7}{10}}$

Counting proof: There are 20 equal probability combinations for the two shown numbers. The following table shows the 20 possible products. The even products are emphasized.

Product	1	2	3	4	5
1	1	**2**	3	**4**	5
2	**2**	**4**	**6**	**8**	**10**
3	3	**6**	9	**12**	15
4	**4**	**8**	**12**	**16**	**20**

Out of these 20 products, 14 are even and 6 are odd. Therefore, the probability that the product of the two shown numbers is even is

$$\frac{14}{20} = \mathbf{\frac{7}{10}}$$

Probability proof: Let B be the event that the product is even. By definition, $p(\overline{B})$ denotes the probability that the product is odd. This happens if and only if both shown numbers are odd. The probability that the 5-sided dice shows an odd number is $3/5$ while the probability that the 4-sided dice shows an odd number is $1/2$. Therefore,

$$p(B) = 1 - p(\overline{B}) = 1 - \frac{3}{5} \cdot \frac{1}{2} = 1 - \frac{3}{10} = \mathbf{\frac{7}{10}}$$

(c) What is the probability that the product of the two shown numbers is even given that their sum is even?

Answer: $\mathbf{\frac{2}{5}}$

Counting proof: There are 20 equal probability combinations for the two shown numbers. The following table shows these 20 possible pairs of sums and products. The 10 pairs with even sums are emphasized.

$(\mathbf{Sum}, \mathbf{Product})$	1	2	3	4	5
1	$(\mathbf{2}, \mathbf{1})$	$(3, 2)$	$(\mathbf{4}, \mathbf{3})$	$(5, 4)$	$(\mathbf{6}, \mathbf{5})$
2	$(3, 2)$	$(\mathbf{4}, \mathbf{4})$	$(5, 6)$	$(\mathbf{6}, \mathbf{8})$	$(7, 10)$
3	$(\mathbf{4}, \mathbf{3})$	$(5, 6)$	$(\mathbf{6}, \mathbf{9})$	$(7, 12)$	$(\mathbf{8}, \mathbf{15})$
4	$(5, 4)$	$(\mathbf{6}, \mathbf{8})$	$(7, 12)$	$(\mathbf{8}, \mathbf{16})$	$(9, 20)$

Out of the 10 combinations that show an even sum, 4 show an even product. Therefore, the probability that the product of the two shown numbers is even given that their sum is even is

$$\frac{4}{10} = \mathbf{\frac{2}{5}}$$

Probability proof: Let A be the event that the sum is even and let B be the event that the product is even. Part (a) proved that

$$p(A) = \frac{1}{2}$$

By definition, $p(A \cap B)$ denotes the probability that both the sum and the product are even. This happens if and only if both shown numbers are even. Otherwise either the sum is odd or the product is odd or both are odd. The probability that the 5-sided dice shows an even number is $2/5$ while the probability that the 4-sided dice shows an even number is $1/2$. Therefore,

$$p(A \cap B) = \frac{2}{5} \cdot \frac{1}{2} = \frac{1}{5}$$

By definition, $p(B|A)$ denotes the probability that the product is even given that the sum is even. Bayes' Theorem implies that

$$p(B|A) = \frac{p(A \cap B)}{p(A)} = \frac{1/5}{1/2} = \mathbf{\frac{2}{5}}$$

(d) What is the probability that the **sum** of the two shown numbers is **even** given that their **product** is **even**?

Answer: $\mathbf{\frac{2}{7}}$

Counting proof: There are 20 equal probability combinations for the two number shown by the two dice. The following table shows these 20 possible pairs of products and sums. The 14 pairs with even products are emphasized.

$(\textbf{Product}, \textbf{Sum})$	1	2	3	4	5
1	$(1, 2)$	$(\mathbf{2}, \mathbf{3})$	$(3, 4)$	$(\mathbf{4}, \mathbf{5})$	$(5, 6)$
2	$(\mathbf{2}, \mathbf{3})$	$(\mathbf{4}, \mathbf{4})$	$(\mathbf{6}, \mathbf{5})$	$(\mathbf{8}, \mathbf{6})$	$(\mathbf{10}, \mathbf{7})$
3	$(3, 4)$	$(\mathbf{6}, \mathbf{5})$	$(9, 6)$	$(\mathbf{12}, \mathbf{7})$	$(15, 8)$
4	$(\mathbf{4}, \mathbf{5})$	$(\mathbf{8}, \mathbf{6})$	$(\mathbf{12}, \mathbf{7})$	$(\mathbf{16}, \mathbf{8})$	$(\mathbf{20}, \mathbf{9})$

Out of the 14 combinations that show an even product, 4 show an even sum. Therefore, the probability that the sum of the two shown numbers is even given that their product is even is

$$\frac{4}{14} = \mathbf{\frac{2}{7}}$$

Probability proof: Let A be the event that the sum is even and let B be the event that the product is even. Part (b) proved that

$$p(B) = \frac{7}{10}$$

By definition, $p(A \cap B)$ denotes the probability that both the sum and the product are even. This happens if and only if both shown numbers are even. Otherwise either the sum is odd or the product is odd or both are odd. The probability that the 5-sided dice shows an even number is $2/5$ while the probability that the 4-sided dice shows an even number is $1/2$. Therefore,

$$p(A \cap B) = \frac{2}{5} \cdot \frac{1}{2} = \frac{1}{5}$$

By definition, $p(A|B)$ denotes the probability that the sum is even given that the product is even. Bayes' Theorem implies that

$$p(A|B) = \frac{p(A \cap B)}{p(B)} = \frac{1/5}{7/10} = \frac{10}{35} = \mathbf{\frac{2}{7}}$$