CISC 2210 TR2– Introduction to Discrete Structures

Midterm Exam 2 – Solutions

April 18, 2023

1. Prove the correctness of the following identity for any $n \geq 1$.

$$\sum_{i=1}^{n}(i-1) + \sum_{i=1}^{n}(i+1) \;=\; n(n+1)$$

Proof by induction:

- *Notations.*

$$\begin{aligned} L(n) &= \sum_{i=1}^{n}(i-1) + \sum_{i=1}^{n}(i+1) \\ R(n) &= n(n+1) \end{aligned}$$

- *Induction base.* Prove that $L(1) = R(1)$:

$$L(1) = (1-1) + (1+1) = 0 + 2 = 2 = 1 \cdot 2 = R(1)$$

- *Induction hypothesis.* Assume that $L(k) = R(k)$ for $k \geq 1$:

$$\sum_{i=1}^{k}(i-1) + \sum_{i=1}^{k}(i+1) \;=\; k(k+1)$$

- *Inductive step.* Prove that $L(k+1) = R(k+1)$ for $k \geq 1$:

$$\begin{aligned} L(k+1) &= \sum_{i=1}^{k+1}(i-1) + \sum_{i=1}^{k+1}(i+1) \\ &= \left(\sum_{i=1}^{k}(i-1) + ((k+1)-1)\right) + \left(\sum_{i=1}^{k}(i+1) + ((k+1)+1)\right) \\ &= \left(\sum_{i=1}^{k}(i-1) + \sum_{i=1}^{k}(i+1)\right) + (k + (k+2)) \\ &= L(k) + (k+k+2) \\ &= R(k) + (2k+2) \\ &= k(k+1) + (2k+2) \\ &= k^2 + k + 2k + 2 \\ &= k^2 + 3k + 2 \\ &= (k+1)(k+2) \\ &= R(k+1) \end{aligned}$$

A proof without induction:

$$\begin{aligned} \sum_{i=1}^{n}(i-1) + \sum_{i=1}^{n}(i+1) &= \sum_{i=1}^{n}((i-1)+(i+1)) \\ &= \sum_{i=1}^{n}(2i) \\ &= 2\sum_{i=1}^{n} i \\ &= 2 \cdot \frac{n(n+1)}{2} \\ &= n(n+1) \end{aligned}$$

2. Consider the following recurrence for integers $n \geq 1$:

$$M(n) = \begin{cases} 2 & \text{for } n = 1 \\ 3M(n-1) - 1 & \text{for } n \geq 2 \end{cases}$$

Prove that for $n \geq 1$

$$M(n) = \frac{3^n + 1}{2}$$

Proof by induction:

- *Induction base.* For $n = 1$

$$M(1) = \frac{3^1 + 1}{2} = \frac{3 + 1}{2} = \frac{4}{2} = 2$$

- *Induction hypothesis.* Assume that for $n > 1$

$$M(n-1) = \frac{3^{n-1} + 1}{2}$$

- *Inductive step.* Prove that for $n > 1$

$$M(n) = \frac{3^n + 1}{2}$$

$$\begin{aligned} M(n) &= 3M(n-1) - 1 \\ &= 3 \cdot \frac{3^{n-1} + 1}{2} - 1 \\ &= \frac{3(3^{n-1} + 1)}{2} - \frac{2}{2} \\ &= \frac{3^n + 3 - 2}{2} \\ &= \frac{3^n + 1}{2} \end{aligned}$$

3. A box contains some blue socks, some green socks, and some red socks.

(a) There are 4 blue socks, 3 green socks, and 2 red socks in the box.

i. Find the number of ways 2 socks can be drawn from the box.

Answer: All together there are $9 = 4 + 3 + 2$ socks. Therefore, the number of ways 2 socks can be drawn from the box is

$$\binom{9}{2} = \frac{9 \cdot 8}{2} = \frac{72}{2} = \mathbf{36}$$

ii. Find the number of ways 2 socks of the same color can be drawn from the box.

Answer: There are $\binom{4}{2}$ possible pairs of blue socks, $\binom{3}{2}$ possible pairs of green socks, and $\binom{2}{2}$ possible pairs of red socks. Therefore, the number of ways 2 socks of the same color can be drawn from the box is

$$\binom{4}{2} + \binom{3}{2} + \binom{2}{2} = \frac{4 \cdot 3}{2} + \frac{3 \cdot 2}{2} + \frac{2 \cdot 1}{2} = \frac{12}{2} + \frac{6}{2} + \frac{2}{2} = 6 + 3 + 1 = \mathbf{10}$$

iii. Find the number of ways 2 socks of different colors can be drawn from the box.

Answer: There are $4 \cdot 3$ blue-green pairs of socks, $4 \cdot 2$ blue-red pairs of socks, and $3 \cdot 2$ green-red pairs of socks. Therefore, the number of ways 2 socks of different colors can be drawn from the box is

$$4 \cdot 3 + 4 \cdot 2 + 3 \cdot 2 = 12 + 8 + 6 = \mathbf{26}$$

iv. How do the three questions above relate to each other?

Answer: A pair of socks is either of the same color or of different colors. Therefore, the answer in part (ii) plus the answer in part (iii) must be equal to the answer in part (i). Indeed,

$$\mathbf{36} = \binom{9}{2} = \left(\binom{4}{2} + \binom{3}{2} + \binom{2}{2}\right) + (4 \cdot 3 + 4 \cdot 2 + 3 \cdot 2) = \mathbf{10} + \mathbf{26}$$

(b) There are n socks in the box out of which b are blue socks, g are green socks, and r are red socks for some three integers $b \geq 2$, $g \geq 2$, and $r \geq 2$ such that $n = b + g + r$.

i. Find the number of ways 2 socks can be drawn from the box.

Answer: All together there are $n = b + g + r$ socks. Therefore, the number of ways 2 socks can be drawn from the box is

$$\binom{b+g+r}{2} = \binom{n}{2} = \frac{n(n-1)}{2}$$

ii. Find the number of ways 2 socks of the same color can be drawn from the box.

Answer: There are $\binom{b}{2}$ possible pairs of blue socks, $\binom{g}{2}$ possible pairs of green socks, and $\binom{r}{2}$ possible pairs of red socks. Therefore, the number of ways 2 socks of the same color can be drawn from the box is

$$\binom{b}{2} + \binom{g}{2} + \binom{r}{2} = \frac{b(b-1)}{2} + \frac{g(g-1)}{2} + \frac{r(r-1)}{2}$$

iii. Find the number of ways 2 socks of different colors can be drawn from the box.

Answer: There are $b \cdot g$ blue-green pairs of socks, $b \cdot r$ blue-red pairs of socks, and $g \cdot r$ green-red pairs of socks. Therefore, the number of ways 2 socks of different colors can be drawn from the box is

$$b \cdot g + b \cdot r + g \cdot r$$

iv. How do the three questions above relate to each other?

Answer: A pair of socks is either of the same color or of different colors. Therefore, the answer in part (ii) plus the answer in part (iii) must be equal to the answer in part (i). Indeed,

$$\begin{aligned}
\binom{b+g+r}{2} &= \frac{(b+g+r)(b+g+r-1)}{2} \\
&= \frac{b(b-1) + g(g-1) + r(r-1) + b(g+r) + g(b+r) + r(b+g)}{2} \\
&= \frac{b(b-1)}{2} + \frac{g(g-1)}{2} + \frac{r(r-1)}{2} + \frac{b \cdot g + b \cdot r + g \cdot b + g \cdot r + r \cdot b + r \cdot g}{2} \\
&= \binom{b}{2} + \binom{g}{2} + \binom{r}{2} + \frac{2(b \cdot g + b \cdot r + g \cdot r)}{2} \\
&= \binom{b}{2} + \binom{g}{2} + \binom{r}{2} + (b \cdot g + b \cdot r + g \cdot r)
\end{aligned}$$

4. Simplify the following expression into an expression that does not contain binomial coefficients, factorials, and fractions.

$$\binom{n+2}{2} - \binom{n}{2}$$

Answer:

$$\binom{n+2}{2} - \binom{n}{2} = \mathbf{2n+1}$$

Proof 1: The formula for $\binom{n}{2}$ implies that $\binom{n+2}{2} = \frac{(n+2)(n+1)}{2}$ and that $\binom{n}{2} = \frac{n(n-1)}{2}$. Therefore

$$\begin{aligned}
\binom{n+2}{2} - \binom{n}{2} &= \frac{(n+2)(n+1)}{2} - \frac{n(n-1)}{2} \\
&= \frac{n^2+3n+2}{2} - \frac{n^2-n}{2} \\
&= \frac{n^2+3n+2-n^2+n}{2} \\
&= \frac{4n+2}{2} \\
&= \frac{2(2n+1)}{2} \\
&= \mathbf{2n+1}
\end{aligned}$$

Proof 2: The recursive formula for binomial coefficients implies that $\binom{n+2}{2} = \binom{n+1}{1} + \binom{n+1}{2}$ and that $\binom{n}{2} = \binom{n+1}{2} - \binom{n}{1}$. Therefore

$$\begin{aligned}
\binom{n+2}{2} - \binom{n}{2} &= \left(\binom{n+1}{1} + \binom{n+1}{2}\right) - \left(\binom{n+1}{2} - \binom{n}{1}\right) \\
&= \binom{n+1}{1} + \binom{n+1}{2} - \binom{n+1}{2} + \binom{n}{1} \\
&= \binom{n+1}{1} + \binom{n}{1} \\
&= (n+1)+n \\
&= \mathbf{2n+1}
\end{aligned}$$

5. Three **fair** Heads (H) and Tails (T) coins are flipped.

   (a) What is the probability that all the coins show Heads?

   Answer: $\mathbf{\frac{1}{8}}$

   Counting proof: There are 8 equal probability combinations for flipping three coins:

   HHH HHT HTH HTT THH THT TTH TTT

   Out of these 8 combinations only **HHH** shows Heads on all three coins. Therefore, the probability that all the coins show Heads is **1/8**.

   Probability proof: Let A be the event that all three coins show Heads. Since the probability of one coin showing Heads is 1/2 and the outcomes of the three flips are independent, it follows that

$$p(A) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \mathbf{\frac{1}{8}}$$

   (b) What is the probability that at least one coin shows Tails?

   Answer: $\mathbf{\frac{7}{8}}$

   Counting proof: There are 8 equal probability combinations for flipping three coins:

   HHH **HHT HTH HTT THH THT TTH TTT**

   Out of these 8 combinations only HHH does not show Heads on all three coins while each one of the other 7 combinations shows at least one Tails. Therefore, the probability that at least one coin shows Tails is **7/8**.

   Probability proof: Let B be the event that at least one coin shows Tails. Since $B = \overline{A}$, it follows that

$$p(B) = 1 - p(A) = 1 - \frac{1}{8} = \mathbf{\frac{7}{8}}$$

(c) What is the probability that all the coins show Heads given that at least one of the coins shows Heads?

Answer: $\mathbf{\frac{1}{7}}$

Counting proof: There are 7 equal probability combinations for flipping three coins in which at least one coin shows Heads:

HHH HHT HTH HTT THH THT TTH

Out of these 7 combinations only **HHH** shows Heads on all three coins. Therefore, the probability that all the coins show Heads is **1/7**.

Probability proof: Let C be the event that at least one of the coins shows Heads. By symmetry,

$$p(C) = p(B) = 7/8$$

Recall that A is the event that all three coins show Heads. Since A is a sub-event of C, it follows that

$$p(A \cap C) = p(A) = \frac{1}{8}$$

By definition, $p(A|C)$ denotes the probability that all the coins show Heads given that at least one of the coins shows Heads. Bayes' Theorem implies that

$$p(A|C) = \frac{p(A \cap C)}{p(C)} = \frac{1/8}{7/8} = \mathbf{\frac{1}{7}}$$

(d) What is the probability that at least one coin shows Tails given that at least one of the coins shows Heads?

Answer: $\mathbf{\frac{6}{7}}$

Counting proof: There are 7 equal probability combinations for flipping three coins in which at least one coin shows Heads:

HHH **HHT HTH HTT THH THT TTH**

Out of these 7 combinations only **HHH** does not show Tails while each one of the other 6 combinations shows at least one Tails. Therefore, at least one coin shows Tails given that at least one of the coins shows Heads is **6/7**.

Probability proof: Let C be the event that at least one of the coins shows Heads. By symmetry,

$$p(C) = p(B) = \frac{7}{8}$$

Recall that B is the event that at least one coin shows Tails. Since the event $B \cap C$ contains the 6 combinations $\mathrm{HHT}, \mathrm{HTH}, \mathrm{HTT}, \mathrm{THH}, \mathrm{THT}, \mathrm{TTH}$ out of the possible 8 combinations, it follows that

$$p(B \cap C) = \frac{6}{8} = \frac{3}{4}$$

By definition, $p(B|C)$ denotes the probability that at least one coin shows Tails given that at least one of the coins shows Heads. Bayes' Theorem implies that

$$p(B|C) = \frac{p(B \cap C)}{p(C)} = \frac{3/4}{7/8} = \frac{24}{28} = \mathbf{\frac{6}{7}}$$