CISC 2210 (TR11) – Introduction to Discrete Structures

Midterm 1 Exam – Solutions

March 3, 2026

1. **Theorem:** For any three sets A, B, and C,

$$C \setminus (A \cap B) = (C \setminus A) \cup (C \setminus B)$$

Proof:

- Let $x \in C \setminus (A \cap B)$. By definition, $x \in C$ and $x \notin (A \cap B)$. The condition $x \notin (A \cap B)$ implies that $x \notin A$ or $x \notin B$. If $x \notin A$, then $x \in C \setminus A$ and if $x \notin B$, then $x \in C \setminus B$. In either case, $x \in (C \setminus A) \cup (C \setminus B)$. This establishes that:

$$C \setminus (A \cap B) \subseteq (C \setminus A) \cup (C \setminus B)$$

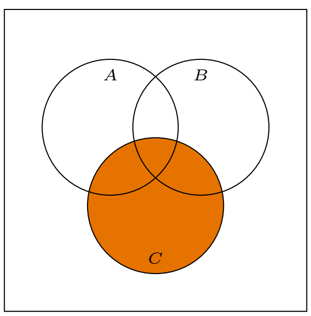
- Let $x \in (C \setminus A) \cup (C \setminus B)$. By definition, $x \in C \setminus A$ or $x \in C \setminus B$. In both cases, it follows that $x \in C$. Furthermore, $x \notin A$ or $x \notin B$ which is equivalent to $x \notin (A \cap B)$. Since $x \in C$ and $x \notin (A \cap B)$, it follows that $x \in C \setminus (A \cap B)$. This establishes that:

$$(C \setminus A) \cup (C \setminus B) \subseteq C \setminus (A \cap B)$$

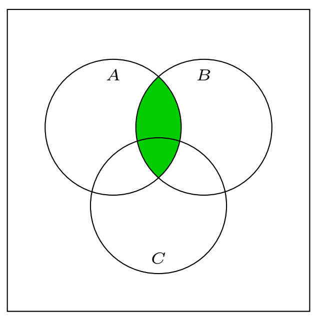
- For any two sets S and T, if $S \subseteq T$ and $T \subseteq S$, then $S = T$. Therefore, for $S = C \setminus (A \cap B)$ and $T = (C \setminus A) \cup (C \setminus B)$, the above arguments show that:

$$C \setminus (A \cap B) = (C \setminus A) \cup (C \setminus B)$$

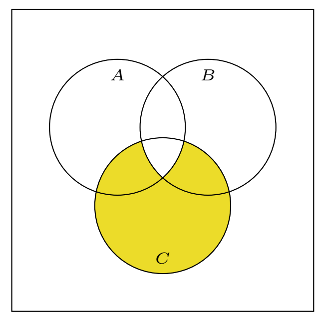
Proof by Venn Diagrams:



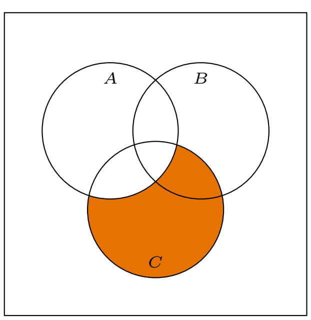
C



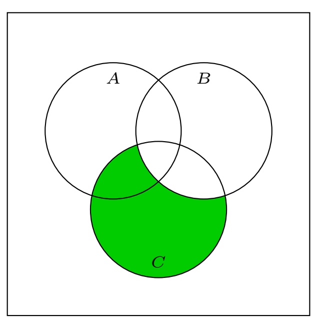
$A \cap B$



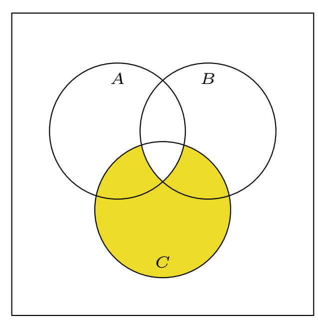
$C \setminus (A \cap B)$



$C \setminus A$



$C \setminus B$

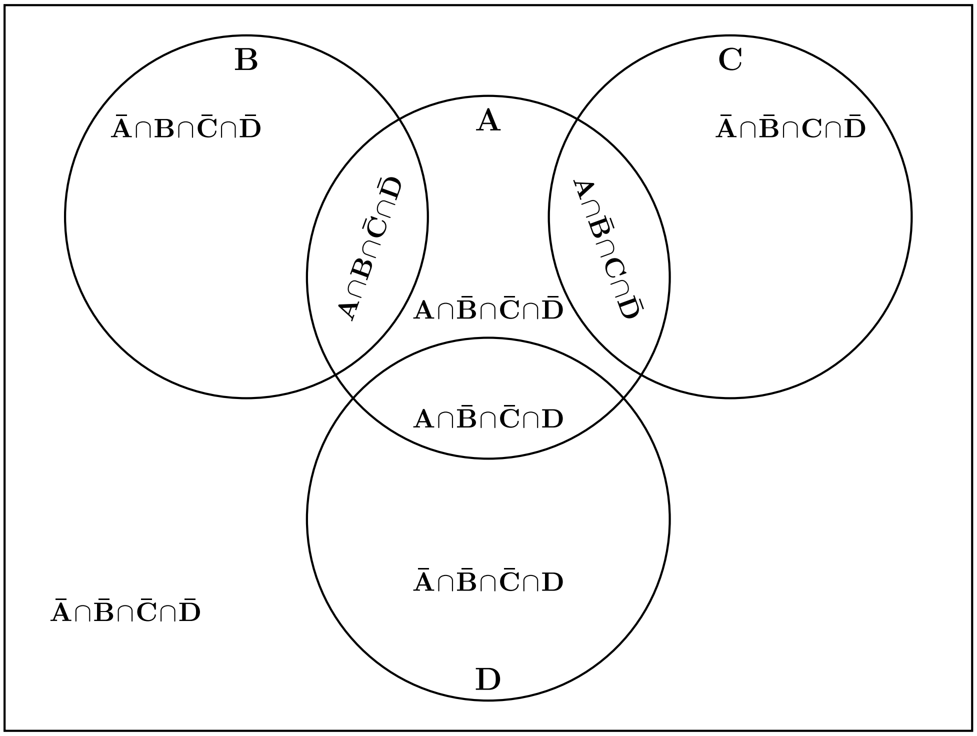


$(C \setminus A) \cup (C \setminus B)$

Remark: This theorem presents a generalized form of the $\overline{A \cap B} = \overline{A} \cup \overline{B}$ De Morgan's Law. While the standard law is defined relative to a universal set $\mathcal{U}$ containing $A \cup B$ and therefore is equivalent to $\mathcal{U} \setminus (A \cap B) = (\mathcal{U} \setminus A) \cup (\mathcal{U} \setminus B)$, the identity in the theorem demonstrates that the relationship remains valid for any arbitrary set C, regardless of whether C contains the union of A and B.

2. The provided Venn diagram illustrates four sets: $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$. A full Venn diagram of four sets defines 16 distinct zones based on set membership. Your goal is to identify which of these 16 potential zones are represented in this specific diagram. For each zone present, clearly indicate its corresponding location in the figure.

a. The following labels identify the eight distinct zones represented in the diagram:



b. There are eight potential zones that are not represented in the diagram:

- The intersection of all four sets: $\mathbf{A}\cap\mathbf{B}\cap\mathbf{C}\cap\mathbf{D}$.
- The four zones representing the intersection of exactly three sets: $\bar{\mathbf{A}}\cap\mathbf{B}\cap\mathbf{C}\cap\mathbf{D}$, $\mathbf{A}\cap\bar{\mathbf{B}}\cap\mathbf{C}\cap\mathbf{D}$, $\mathbf{A}\cap\mathbf{B}\cap\bar{\mathbf{C}}\cap\mathbf{D}$, and $\mathbf{A}\cap\mathbf{B}\cap\mathbf{C}\cap\bar{\mathbf{D}}$.
- The three possible pairwise intersections among the sets $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ involving the complements of the remaining sets: $\bar{\mathbf{A}}\cap\mathbf{B}\cap\mathbf{C}\cap\bar{\mathbf{D}}$, $\bar{\mathbf{A}}\cap\mathbf{B}\cap\bar{\mathbf{C}}\cap\mathbf{D}$, and $\bar{\mathbf{A}}\cap\bar{\mathbf{B}}\cap\mathbf{C}\cap\mathbf{D}$.

3. At a dog show, participants could earn ribbons in five colors: Blue, Red, Green, Yellow, and Purple. Based on the following data, determine the total number of dogs that attended the show:
   - Only one dog wore no ribbons.
   - 3 dogs wore all five ribbons.
   - 2 dogs wore Blue and Red ribbons, but did not wear Green, Yellow, or Purple ribbons.
   - The rest of the dogs wore exactly one ribbon.
   - The total counts for each ribbon color were:
     - 9 dogs wore a Blue ribbon.
     - 8 dogs wore a Red ribbon.
     - 7 dogs wore a Green ribbon.
     - 6 dogs wore a Yellow ribbon.
     - 5 dogs wore a Purple ribbon.

Answer: **22** dogs attended the show.

Explanation I: The total number of ribbons awarded at the show was:

$$9 + 8 + 7 + 6 + 5 = 35$$

However, the number of dogs wearing ribbons is less than 35 because some dogs received multiple awards. To determine the actual number of dogs wearing ribbons, the total ribbon count must be adjusted by subtracting the "extra" ribbons:

- Each of the three dogs wearing five ribbons accounts for 4 "extra" ribbons beyond the first $(5 - 1 = 4)$. This requires a subtraction of $4 \cdot 3 = 12$.
- Each of the two dogs wearing two ribbons accounts for 1 "extra" ribbon beyond the first $(2 - 1 = 1)$. This requires a subtraction of $1 \cdot 2 = 2$.

The number of dogs wearing at least one ribbon is therefore:

$$35 - 12 - 2 = 21$$

Including the one dog that wore no ribbons, the total number of dogs that attended the show was:

$$\mathbf{22} = 21 + 1$$

Explanation II: For $k \in \{0, 1, 2, 3, 4, 5\}$, let S_k denote the set of dogs wearing exactly k ribbons and let $s_k = |S_k|$. The total number of dogs in attendance, denoted by D, is defined as the sum of the sizes of these six disjoint sets:

$$D = s_0 + s_1 + s_2 + s_3 + s_4 + s_5$$

The first three data items establish that $s_0 = 1$, $s_5 = 3$, and $s_2 = 2$. The fourth data item implies that $s_3 = s_4 = 0$. Substituting these values into the expression for D yields:

$$D = 1 + s_1 + 2 + 0 + 0 + 3 = s_1 + 6$$

There are two methods to calculate the total number of awarded ribbons, denoted by R. The first is to sum the individual counts provided in the fifth data item:

$$R = 9 + 8 + 7 + 6 + 5 = 35$$

The second method expresses R as a weighted sum of the s_k variables:

$$R = (1 \cdot s_1) + (2 \cdot s_2) + (3 \cdot s_3) + (4 \cdot s_4) + (5 \cdot s_5) = s_1 + (2 \cdot 2) + (3 \cdot 0) + (4 \cdot 0) + (5 \cdot 3) = s_1 + 19$$

Equating these two expressions for R allows for the calculation of s_1:

$$s_1 + 19 = 35 \implies s_1 = 16$$

Substituting $s_1 = 16$ back into the equation for D gives the total attendance:

$$D = s_1 + 6 = 16 + 6 = \mathbf{22}$$

4. **Theorem:** For any four boolean variables x, y, z, and w, the following identity holds:

$$(x \vee y) \wedge (y \vee z) \wedge (z \vee w) \wedge (w \vee x) \equiv (x \wedge z) \vee (y \wedge w)$$

Proof: The chain of identities below demonstrates that the right-hand side and the left-hand side of the above identity are equivalent: (i) In the first step, the distributive law $P \vee (Q \wedge R) = (P \vee Q) \wedge (P \vee R)$ is applied, (ii) in the second step, the distributive law $(Q \wedge R) \vee P = (Q \vee P) \wedge (R \vee P)$ is applied twice, and (iii) in the third step, the commutative law is applied several times.

$$\begin{aligned}(x \wedge z) \vee (y \wedge w) &\equiv ((x \wedge z) \vee y) \wedge ((x \wedge z) \vee w)\\ &\equiv (x \vee y) \wedge (z \vee y) \wedge (x \vee w) \wedge (z \vee w)\\ &\equiv (x \vee y) \wedge (y \vee z) \wedge (z \vee w) \wedge (w \vee x)\end{aligned}$$

Proof using truth tables: The identity is correct because the rightmost column of the top table (representing the left-hand side of the above identity) and the rightmost column of the bottom table (representing the right-hand side of the above identity) are identical.

x	y	z	w	$x \vee y$	$y \vee z$	$z \vee w$	$w \vee x$	$(x \vee y) \wedge (y \vee z) \wedge (z \vee w) \wedge (w \vee x)$
T	T	T	T	T	T	T	T	T
T	T	T	F	T	T	T	T	T
T	T	F	T	T	T	T	T	T
T	T	F	F	T	T	F	T	F
T	F	T	T	T	T	T	T	T
T	F	T	F	T	T	T	T	T
T	F	F	T	T	F	T	T	F
T	F	F	F	T	F	F	T	F
F	T	T	T	T	T	T	T	T
F	T	T	F	T	T	T	F	F
F	T	F	T	T	T	T	T	T
F	T	F	F	T	T	F	F	F
F	F	T	T	F	T	T	T	F
F	F	T	F	F	T	T	F	F
F	F	F	T	F	F	T	T	F
F	F	F	F	F	F	F	F	F

x	y	z	w	$x \wedge z$	$y \wedge w$	$(x \wedge z) \vee (y \wedge w)$
T	T	T	T	T	T	T
T	T	T	F	T	F	T
T	T	F	T	F	T	T
T	T	F	F	F	F	F
T	F	T	T	T	F	T
T	F	T	F	T	F	T
T	F	F	T	F	F	F
T	F	F	F	F	F	F
F	T	T	T	F	T	T
F	T	T	F	F	F	F
F	T	F	T	F	T	T
F	T	F	F	F	F	F
F	F	T	T	F	F	F
F	F	T	F	F	F	F
F	F	F	T	F	F	F
F	F	F	F	F	F	F

5. For each of the following four logical propositions involving boolean variables x and y, determine whether the statement is a tautology.

- If the proposition is a tautology, provide a formal proof demonstrating that the proposition evaluates to true for all four possible truth assignments to x and y.
- If the proposition is not a tautology, identify at least one truth assignment for x and y (a counterexample) for which the proposition evaluates to false.

A reference truth table is provided below for the two-variable functions used in these propositions: $\mathcal{AND}$ ($\wedge$), $\mathcal{OR}$ ($\vee$), $\mathcal{NAND}$ ($\uparrow$), and $\mathcal{NOR}$ ($\downarrow$). Also provided below is the truth table of the $\mathcal{IMPLY}$ ($\rightarrow$) function of two propositions P and Q.

x	y	$x \wedge y$	$x \vee y$	$x \uparrow y$	$x \downarrow y$
T	T	T	T	F	F
T	F	F	T	T	F
F	T	F	T	T	F
F	F	F	F	T	T

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

a. The proposition $(x \wedge y) \rightarrow (x \vee y)$ is a tautology. As demonstrated in the truth table below, it evaluates to True for all four possible truth assignments to the variables x and y.

x	y	$x \wedge y$	$x \vee y$	$(x \wedge y) \rightarrow (x \vee y)$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

b. The proposition $(x \wedge y) \rightarrow (x \downarrow y)$ is not a tautology. As demonstrated in the truth table below, it evaluates to False when $x = T$ and $y = T$.

x	y	$x \wedge y$	$x \downarrow y$	$(x \wedge y) \rightarrow (x \downarrow y)$
T	T	T	F	F
T	F	F	F	T
F	T	F	F	T
F	F	F	T	T

c. The proposition $(x \uparrow y) \rightarrow (x \vee y)$ is not a tautology. As demonstrated in the truth table below, it evaluates to False when $x = F$ and $y = F$.

x	y	$x \uparrow y$	$x \vee y$	$(x \wedge y) \rightarrow (x \vee y)$
T	T	F	T	T
T	F	T	T	T
F	T	T	T	T
F	F	T	F	F

d. The proposition $(x \uparrow y) \rightarrow (x \downarrow y)$ is not a tautology. As demonstrated in the truth table below, it evaluates to False either when $x = T$ and $y = F$, or when $x = F$ and $y = T$.

x	y	$x \uparrow y$	$x \downarrow y$	$(x \wedge y) \rightarrow (x \downarrow y)$
T	T	F	F	T
T	F	T	F	F
F	T	T	F	F
F	F	T	T	T

6. A card has a statement written on each of its two sides as follows:
   - Side A: "The statement on the other side is true."
   - Side B: "The statement on the other side is false."

   Prove that this situation constitutes a paradox.

   Notation: Denote by S_A and S_B the statements on side A and side B, respectively.

   Proof I: To prove that this situation constitutes a paradox, it must be demonstrated that no consistent truth value can be assigned to the statements on both sides of the card.

   - If S_A is assumed to be True, then S_B must also be True. However, because S_B asserts that S_A is False, this necessitates that S_A is False, creating a contradiction.
   - If S_A is instead assumed to be False, then its claim that S_B is True must be incorrect, meaning that S_B is False. Yet, if S_B is False, its assertion that S_A is False must be a lie, implying that S_A is True, creating a contradiction.

   Since every possible logical starting point for S_A necessitates its own negation, the truth status of the card is inherently self-contradictory and therefore the situation constitutes a paradox.

   Proof II: To prove that this situation constitutes a paradox, it must be demonstrated that no consistent truth value can be assigned to the statements on both sides of the card.

   - If S_B is assumed to be True, then its claim that the S_A is False must be valid. However, in this case S_A's assertion that S_B is True must be incorrect, necessitating that S_B is False, creating a contradiction.
   - Conversely, if S_B is assumed to be False, then its assertion that S_A is False must be incorrect, implying that S_A is True and therefore S_B is also True, creating a contradiction.

   Since every possible logical starting point for S_B necessitates its own negation, the truth status of the card is inherently self-contradictory and therefore the situation constitutes a paradox.