

# CISC 2210 (TR2) – Introduction to Discrete Structures

## Midterm 1 Exam – Solutions

March 3, 2026

1. **Theorem:** For any three sets  $A$ ,  $B$ , and  $C$ ,

$$C \setminus (A \cup B) = (C \setminus A) \cap (C \setminus B)$$

**Proof:**

- Let  $x \in C \setminus (A \cup B)$ . By definition,  $x \in C$  and  $x \notin (A \cup B)$ . The condition  $x \notin (A \cup B)$  implies that  $x \notin A$  and  $x \notin B$ . It follows that  $x \in C \setminus A$  since  $x \in C$  and  $x \notin A$  and  $x \in C \setminus B$  since  $x \in C$  and  $x \notin B$ . Therefore,  $x \in (C \setminus A) \cap (C \setminus B)$ . This establishes that:

$$C \setminus (A \cup B) \subseteq (C \setminus A) \cap (C \setminus B)$$

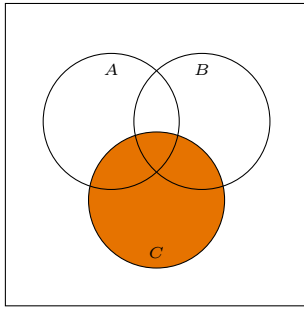
- Let  $x \in (C \setminus A) \cap (C \setminus B)$ . By definition,  $x \in C \setminus A$  and  $x \in C \setminus B$ . This implies that  $x \in C$ ,  $x \notin A$ , and  $x \notin B$ . Since  $x \notin A$  and  $x \notin B$ , it follows that  $x \notin (A \cup B)$ . Therefore  $x \in C \setminus (A \cup B)$  because  $x \in C$ . This establishes that:

$$(C \setminus A) \cap (C \setminus B) \subseteq C \setminus (A \cup B)$$

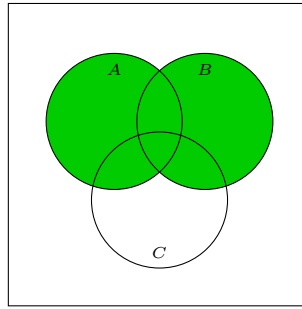
- For any two sets  $S$  and  $T$ , if  $S \subseteq T$  and  $T \subseteq S$ , then  $S = T$ . Therefore, for  $S = C \setminus (A \cup B)$  and  $T = (C \setminus A) \cap (C \setminus B)$ , the above arguments show that:

$$C \setminus (A \cup B) = (C \setminus A) \cap (C \setminus B)$$

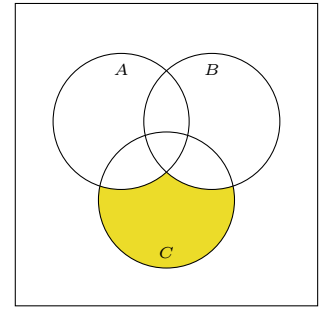
**Proof by Venn Diagrams:**



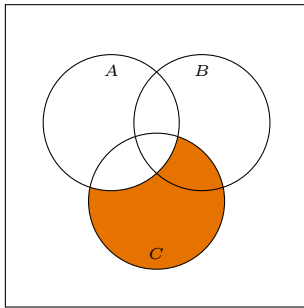
$C$



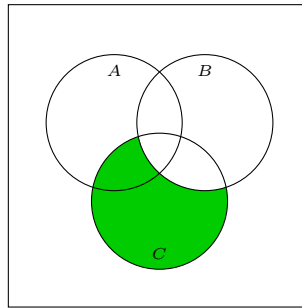
$A \cup B$



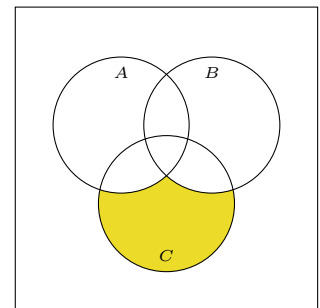
$C \setminus (A \cup B)$



$C \setminus A$



$C \setminus B$

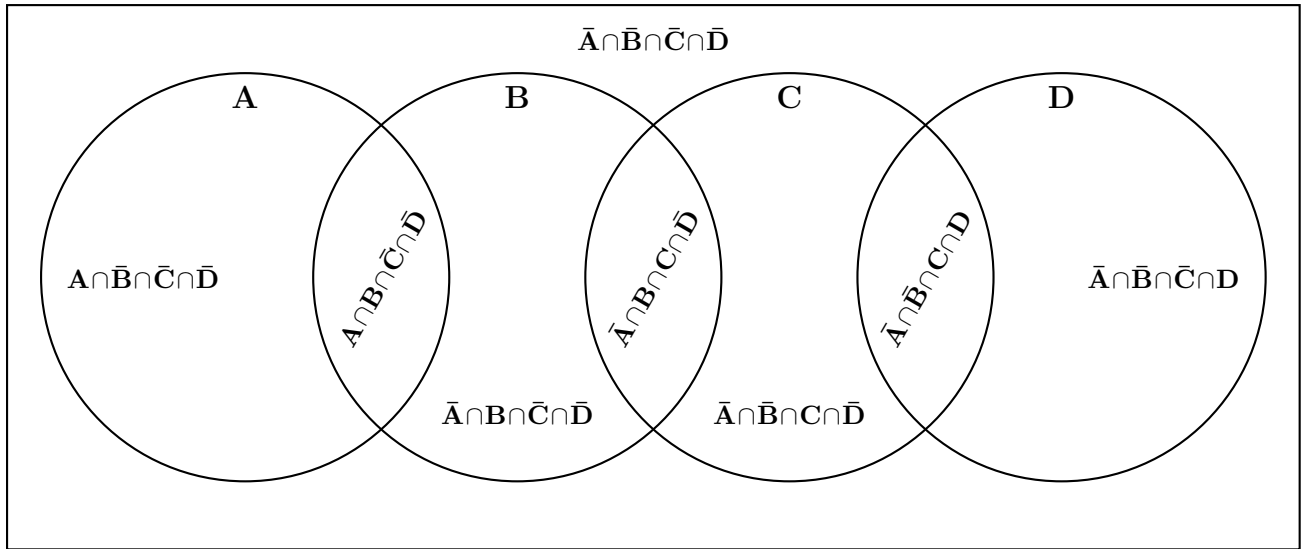


$(C \setminus A) \cap (C \setminus B)$

**Remark:** This theorem presents a generalized form of the  $\overline{A \cup B} = \bar{A} \cap \bar{B}$  De Morgan's Law. While the standard law is defined relative to a universal set  $\mathcal{U}$  containing  $A \cup B$  and therefore is equivalent to  $\mathcal{U} \setminus (A \cup B) = (\mathcal{U} \setminus A) \cap (\mathcal{U} \setminus B)$ , the identity in the theorem demonstrates that the relationship remains valid for any arbitrary set  $C$ , regardless of whether  $C$  contains the union of  $A$  and  $B$ .

2. The provided Venn diagram illustrates four sets: **A, B, C, D**. A full Venn diagram of four sets defines 16 distinct zones based on set membership. Your goal is to identify which of these 16 potential zones are represented in this specific diagram. For each zone present, clearly indicate its corresponding location in the figure.

a. The following labels identify the eight distinct zones represented in the diagram:



b. There are eight potential zones that are not represented in the diagram:

- The intersection of all four sets:  $A \cap B \cap C \cap D$ .
- The four zones representing the intersection of exactly three sets:  $\bar{A} \cap B \cap C \cap D$ ,  $A \cap \bar{B} \cap C \cap D$ ,  $A \cap B \cap \bar{C} \cap D$ , and  $A \cap B \cap C \cap \bar{D}$ .
- Three specific pairwise intersections among two sets involving the complements of the remaining sets:  $A \cap \bar{B} \cap C \cap \bar{D}$ ,  $A \cap \bar{B} \cap \bar{C} \cap D$ , and  $\bar{A} \cap B \cap \bar{C} \cap D$ .

3. At a sports festival, athletes could win medals in five different sports: Basketball, Tennis, Swimming, Running, and Cycling. Based on the following data, determine the total number of athletes who participated in the festival.
- Only one athlete did not win a medal in any of the five sports.
  - Exactly 2 athletes won a medal in all five sports.
  - Exactly 3 athletes won medals in Basketball, Tennis, and Swimming but did not win a medal in Running or Cycling.
  - The rest of the athletes won a medal in exactly one sport.
  - The total counts of medals awarded for each sport were:
    - 12 medals in Basketball.
    - 10 medals in Tennis.
    - 8 medals in Swimming.
    - 6 medals in Running.
    - 4 medals in Cycling.

**Answer:** 27 athletes participated in the festival.

**Explanation I:** The total count of all medals awarded at the festival was:

$$12 + 10 + 8 + 6 + 4 = 40$$

However, the number of athletes who won medals is less than 40 because some athletes received multiple medals. To determine the actual number of athletes who won at least one medal, the total medal count must be adjusted by subtracting the “extra” medals:

- Each of the two athletes who won five medals accounts for 4 “extra” medals beyond the first ( $5 - 1 = 4$ ). This requires a subtraction of  $4 \cdot 2 = 8$ .
- Each of the three athletes who won three medals accounts for 2 “extra” medals beyond the first ( $3 - 1 = 2$ ). This requires a subtraction of  $2 \cdot 3 = 6$ .

The number of athletes who won at least one medal is therefore:

$$40 - 8 - 6 = 26$$

Including the one athlete who won no medals, the total number of athletes who participated in the festival was:

$$27 = 26 + 1$$

**Explanation II:** For  $k \in \{0, 1, 2, 3, 4, 5\}$ , let  $S_k$  denote the set of athletes who won exactly  $k$  medals and let  $s_k = |S_k|$ . The total number of athletes who participated in the festival, denoted by  $A$ , is defined as the sum of the sizes of these six disjoint sets:

$$A = s_0 + s_1 + s_2 + s_3 + s_4 + s_5$$

The first three data items establish that  $s_0 = 1$ ,  $s_5 = 2$ , and  $s_3 = 3$ . The fourth data item implies that  $s_2 = s_4 = 0$ . Substituting these values into the expression for  $A$  yields:

$$A = 1 + s_1 + 0 + 3 + 0 + 2 = s_1 + 6$$

There are two methods to calculate the total number of awarded medals, denoted by  $M$ . The first is to sum the individual counts provided in the fifth data item:

$$M = 12 + 10 + 8 + 6 + 4 = 40$$

The second method expresses  $M$  as a weighted sum of the  $s_k$  variables:

$$M = (1 \cdot s_1) + (2 \cdot s_2) + (3 \cdot s_3) + (4 \cdot s_4) + (5 \cdot s_5) = s_1 + (2 \cdot 0) + (3 \cdot 3) + (4 \cdot 0) + (5 \cdot 2) = s_1 + 19$$

Equating these two expressions for  $M$  allows for the calculation of  $s_1$ :

$$s_1 + 19 = 40 \implies s_1 = 21$$

Substituting  $s_1 = 21$  back into the equation for  $A$  gives the total attendance:

$$A = s_1 + 6 = 21 + 6 = \mathbf{27}$$

4. **Theorem:** For any four boolean variables  $x, y, z$ , and  $w$ , the following identity holds:

$$(\mathbf{x} \wedge \mathbf{y}) \vee (\mathbf{y} \wedge \mathbf{z}) \vee (\mathbf{z} \wedge \mathbf{w}) \vee (\mathbf{w} \wedge \mathbf{x}) \equiv (\mathbf{x} \vee \mathbf{z}) \wedge (\mathbf{y} \vee \mathbf{w})$$

**Proof:** The chain of identities below demonstrates that the right-hand side and the left-hand side of the above identity are equivalent: (i) In the first step, the distributive law  $P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R)$  is applied, (ii) in the second step, the distributive law  $(Q \vee R) \wedge P = (Q \wedge P) \vee (R \wedge P)$  is applied twice, and (iii) in the third step, the commutative law is applied several times.

$$\begin{aligned} (x \vee z) \wedge (y \vee w) &\equiv ((x \vee z) \wedge y) \vee ((x \vee z) \wedge w) \\ &\equiv (x \wedge y) \vee (z \wedge y) \vee (x \wedge w) \vee (z \wedge w) \\ &\equiv (x \wedge y) \vee (y \wedge z) \vee (z \wedge w) \vee (w \wedge x) \end{aligned}$$

**Proof using truth tables:** The identity is correct because the rightmost column of the top table (representing the left-hand side of the above identity) and the rightmost column of the bottom table (representing the right-hand side of the above identity) are identical.

$x$	$y$	$z$	$w$	$x \wedge y$	$y \wedge z$	$z \wedge w$	$w \wedge x$	$(x \wedge y) \vee (y \wedge z) \vee (z \wedge w) \vee (w \wedge x)$
$T$	$T$	$T$	$T$	$T$	$T$	$T$	$T$	$T$
$T$	$T$	$T$	$F$	$T$	$T$	$F$	$F$	$T$
$T$	$T$	$F$	$T$	$T$	$F$	$F$	$T$	$T$
$T$	$T$	$F$	$F$	$T$	$F$	$F$	$F$	$T$
$T$	$F$	$T$	$T$	$F$	$F$	$T$	$T$	$T$
$T$	$F$	$T$	$F$	$F$	$F$	$F$	$F$	$F$
$T$	$F$	$F$	$T$	$F$	$F$	$F$	$T$	$T$
$T$	$F$	$F$	$F$	$F$	$F$	$F$	$F$	$F$
$F$	$T$	$T$	$T$	$F$	$T$	$T$	$F$	$T$
$F$	$T$	$T$	$F$	$F$	$T$	$F$	$F$	$T$
$F$	$T$	$F$	$T$	$F$	$F$	$F$	$F$	$F$
$F$	$T$	$F$	$F$	$F$	$F$	$F$	$F$	$F$
$F$	$F$	$T$	$T$	$F$	$F$	$T$	$F$	$T$
$F$	$F$	$T$	$F$	$F$	$F$	$F$	$F$	$F$
$F$	$F$	$F$	$T$	$F$	$F$	$F$	$F$	$F$
$F$	$F$	$F$	$F$	$F$	$F$	$F$	$F$	$F$

$x$	$y$	$z$	$w$	$x \vee z$	$y \vee w$	$(x \vee z) \wedge (y \vee w)$
$T$	$T$	$T$	$T$	$T$	$T$	$T$
$T$	$T$	$T$	$F$	$T$	$T$	$T$
$T$	$T$	$F$	$T$	$T$	$T$	$T$
$T$	$T$	$F$	$F$	$T$	$T$	$T$
$T$	$F$	$T$	$T$	$T$	$T$	$T$
$T$	$F$	$T$	$F$	$T$	$F$	$F$
$T$	$F$	$F$	$T$	$T$	$T$	$T$
$T$	$F$	$F$	$F$	$T$	$F$	$F$
$F$	$T$	$T$	$T$	$T$	$T$	$T$
$F$	$T$	$T$	$F$	$T$	$T$	$T$
$F$	$T$	$F$	$T$	$F$	$T$	$F$
$F$	$T$	$F$	$F$	$F$	$T$	$F$
$F$	$F$	$T$	$T$	$T$	$T$	$T$
$F$	$F$	$T$	$F$	$T$	$F$	$F$
$F$	$F$	$F$	$T$	$F$	$T$	$F$
$F$	$F$	$F$	$F$	$F$	$F$	$F$

5. For each of the following four logical propositions involving boolean variables  $x$  and  $y$ , determine whether the statement is a tautology.

- If the proposition is a tautology, provide a formal proof demonstrating that the proposition evaluates to true for all four possible truth assignments to  $x$  and  $y$ .
- If the proposition is not a tautology, identify at least one truth assignment for  $x$  and  $y$  (a counterexample) for which the proposition evaluates to false.

A reference truth table is provided below for the two-variable functions used in these propositions:  $\mathcal{AND}$  ( $\wedge$ ),  $\mathcal{OR}$  ( $\vee$ ),  $\mathcal{NAND}$  ( $\uparrow$ ), and  $\mathcal{NOR}$  ( $\downarrow$ ). Also provided below is the truth table of the  $\mathcal{IMPLY}$  ( $\rightarrow$ ) function of two propositions  $P$  and  $Q$ .

$x$	$y$	$x \wedge y$	$x \vee y$	$x \uparrow y$	$x \downarrow y$
$T$	$T$	$T$	$T$	$F$	$F$
$T$	$F$	$F$	$T$	$T$	$F$
$F$	$T$	$F$	$T$	$T$	$F$
$F$	$F$	$F$	$F$	$T$	$T$

$P$	$Q$	$P \rightarrow Q$
$T$	$T$	$T$
$T$	$F$	$F$
$F$	$T$	$T$
$F$	$F$	$T$

a. The proposition  $(x \vee y) \rightarrow (x \wedge y)$  is not a tautology. As demonstrated in the truth table below, it evaluates to False either when  $x = T$  and  $y = F$ , or when  $x = F$  and  $y = T$ .

$x$	$y$	$x \vee y$	$x \wedge y$	$(x \vee y) \rightarrow (x \wedge y)$
$T$	$T$	$T$	$T$	$T$
$T$	$F$	$T$	$F$	$F$
$F$	$T$	$T$	$F$	$F$
$F$	$F$	$F$	$F$	$T$

b. The proposition  $(x \vee y) \rightarrow (x \uparrow y)$  is not a tautology. As demonstrated in the truth table below, it evaluates to False when  $x = T$  and  $y = T$ .

$x$	$y$	$x \vee y$	$x \uparrow y$	$(x \vee y) \rightarrow (x \uparrow y)$
$T$	$T$	$T$	$F$	$F$
$T$	$F$	$T$	$T$	$T$
$F$	$T$	$T$	$T$	$T$
$F$	$F$	$F$	$T$	$T$

c. The proposition  $(x \downarrow y) \rightarrow (x \wedge y)$  is not a tautology. As demonstrated in the truth table below, it evaluates to False when  $x = F$  and  $y = F$ .

$x$	$y$	$x \downarrow y$	$x \wedge y$	$(x \downarrow y) \rightarrow (x \wedge y)$
$T$	$T$	$F$	$T$	$T$
$T$	$F$	$F$	$F$	$T$
$F$	$T$	$F$	$F$	$T$
$F$	$F$	$T$	$F$	$F$

d. The proposition  $(x \downarrow y) \rightarrow (x \uparrow y)$  is a tautology. As demonstrated in the truth table below, it evaluates to True for all four possible truth assignments to the variables  $x$  and  $y$ .

$x$	$y$	$x \downarrow y$	$x \uparrow y$	$(x \downarrow y) \rightarrow (x \uparrow y)$
$T$	$T$	$F$	$F$	$T$
$T$	$F$	$F$	$T$	$T$
$F$	$T$	$F$	$T$	$T$
$F$	$F$	$T$	$T$	$T$

6. A card contains the following statements regarding two distinct integers,  $n$  and  $k$ :

- $n > k$ .
- $k > n$ .
- Exactly one of the three statements on this card (including this one) is true.

Prove that this situation constitutes a paradox.

**Proof I:** To prove that this situation constitutes a paradox, it must be demonstrated that every possible truth assignment for the third statement leads to a logical contradiction.

- If the third statement is assumed to be True, then exactly one of the three statements must be True. However, since  $n$  and  $k$  are distinct integers, either  $n > k$  or  $k > n$  must be True, which would result in a total of two True statements ( $n > k$  and the third statement, or  $k > n$  and the third statement), thereby contradicting the claim that only one is True.
- Conversely, if the third statement is assumed to be False, then its claim that exactly one statement is True must be incorrect, meaning either zero or at least two statements must be True. However, because  $n$  and  $k$  are distinct, exactly one of the first two statements ( $n > k$  or  $k > n$ ) is guaranteed to be True, while the third is already assumed False, resulting in exactly one True statement total which would make the third statement True, contradicting the assumption that it is False.

Because assuming the third statement is True makes it False, and assuming it is False makes it True, the situation is inherently self-contradictory and therefore constitutes a paradox.

**Proof II:** Since  $n$  and  $k$  are distinct integers, it follows that exactly one of the first two statements ( $n > k$  or  $k > n$ ) must be True, while the other is necessarily False.

- If the third statement is assumed to be True, then the total count of true statements on the card would be two (one from the first two statements plus the third statement itself). This contradicts its own claim that only one statement is True.
- Conversely, if the third statement is assumed to be False, then its claim that exactly one statement is True is incorrect. However, because exactly one of the first two statements is True and the third is False, the total count of True statements is exactly one. This result would make the third statement True, contradicting the assumption that it is False.

Because the third statement is False when it is assumed to be True and True when it is assumed to be False, the situation constitutes a paradox.