Discrete Structures: Counting and Combinatorics

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Numbers

How to count to 1000 on two hands

- https://www.youtube.com/watch?v=1SMmc9gQmHQ

“Fascinating” properties of the numbers 1 to 9

- https://www.youtube.com/watch?v=ByZLqOF-Jjk

There are 365 days and 12 months in a year

- $365 = 10^2 + 11^2 + 12^2 = 13^2 + 14^2$.
- It is the only integer for which there exists an integer n such that

$$(n-2)^2 + (n-1)^2 + n^2 = (n+1)^2 + (n+2)^2$$

The Numbers 144 and 441

144 vs. 441

- Both are perfect square: $144 = 12^2$ and $441 = 21^2$.
- 144 is the "reverse" of 441 and 12 is the "reverse" of 21.

There are squares everywhere!

- All of their digits are perfect squares: $1 = 1^2$ and $4 = 2^2$.
- The sum of their digits is a perfect square: $1 + 4 + 4 = 9 = 3^2$.
- The product of their digits is a perfect square: $1 \times 4 \times 4 = 16 = 4^2$.
- The sum of the digits is the square of the number of digits.
- The square of the sum of the digits of their square roots is the sum of the digits.

Can Numbers be Interesting?

The taxi-cub number

- 1729 is the smallest number that is the sum of two cubes in two different ways:

$$1729 = 12^3 + 1^3 = 10^3 + 9^3$$

Story

- This number is called a taxicab number, because in a discussion between the mathematicians G. H. Hardy and Srinivasa Ramanujan about interesting and dull numbers, the former remarked that the number 1729 of the taxicab he had ridden seemed uninteresting, and the latter immediately answered that it is interesting, being the smallest number that is the sum of two cubes in two different ways.

All the Positive Integers Are Interesting!

Proof

- The ”proof” is by contradiction.
- If there exists a non-empty set of uninteresting natural numbers, then there must be a smallest uninteresting number.
- But the smallest uninteresting number is itself interesting because it is the smallest uninteresting number: a contradiction.

Online resources

- https://www.youtube.com/watch?v=Ysd1XhqMbe8
- https://en.m.wikipedia.org/wiki/Interesting_number_paradox

Euler's Identity

The identity

$$e^{\pi i} + 1 = 0 \qquad e^{\pi i} = -1$$

A joke???

π tells i: *get real!*
i answers to π: *be rational!*
e tells both of them: *join me and we will be - one!*

https://www.youtube.com/watch?v=IUTGFQpKaPU

Online resources

- https://www.youtube.com/watch?v=sKtloBAuP74&t=233s
- https://www.youtube.com/watch?v=NXrBoWOBvIY
- https://www.youtube.com/watch?v=-dhHrg-KbJ0

Which is larger e^{π} or π^{e}?

- https://www.youtube.com/watch?v=JE_YeVlSqLY
- https://www.youtube.com/watch?v=I7wiS9rH2h0

Representing Numbers: Decimal, Binary, . . .

Different Bases For Numbers

- Online videos:
  - https://www.youtube.com/watch?v=1srwWeMe3BE
  - https://www.youtube.com/watch?v=aW3qCcH6Dao
  - https://www.youtube.com/watch?v=Fpm-E5v6ddc
- A short tutorial:
  - https://www.tutorialspoint.com/computer_logical_organization/number_system_conversion.htm

The Josephus Problem

- https://www.youtube.com/watch?v=uCsD3ZGzMgE

Combinations and Permutations

Introduction with Cartoon Slides

- http://tinytram.com/math/combinatorics/

Online resources

- https://www.youtube.com/watch?v=uNS1QvDzCVw&feature=youtu.be
- https://www.youtube.com/watch?v=hVqq3nm0IHs
- https://youtu.be/LM5iOHKo_Fc?list=PLMyAzUai9V3ox_LDwl54GRkNxovx6NqQX

Four Counting Types

Setting

- **Input:** A universal set $\mathcal{S} = \{s_1, s_2, \ldots, s_n\}$ with n distinct objects.
- **Goal:** Count the number of some structures with some parameters containing objects from $\mathcal{S}$.

Structures

- **Permutations:** In how many ways can all the objects of $\mathcal{S}$ be ordered?
- **Lists with repetitions:** How many lists with $1 \leq k$ objects from $\mathcal{S}$ are there when repetitions are allowed?
- **Lists without repetitions:** How many lists with $1 \leq k \leq n$ objects from $\mathcal{S}$ are ther when repetitions are not allowed?
- **Subsets:** How many subsets of $\mathcal{S}$ of size $1 \leq k \leq n$ are there?

Example: $\mathcal{S} = \{R, B, G, M\}$

There are $3! = 6$ **permutations of** $\{R, B, G\}$

$RBG \quad RGB \quad BRG \quad BGR \quad GRB \quad GBR$

There are $4! = 24$ **permutations of** $\mathcal{S} = \{R, B, G, M\}$

$RBGM$	$RBMG$	$RGBM$	$RGMB$	$RMBG$	$RMGB$
$BRGM$	$BRMG$	$BGRM$	$BGMR$	$BMRG$	$BMGR$
$GRBM$	$GRMB$	$GBRM$	$GBMR$	$GMRB$	$GMBR$
$MRBG$	$MRGB$	$MBRG$	$MBGR$	$MGRB$	$MGBR$

Example: $\mathcal{S} = \{R, B, G, M\}$

There are $4^2 = 16$ lists with repetitions of length 2

$$\begin{array}{cccc} RR & RB & RG & RM \\ BR & BB & BG & BM \\ GR & GB & GG & GM \\ MR & MB & MG & MM \end{array}$$

There are $4^4 = 256$ lists with repetitions of length 4

$$\begin{array}{cccccc} & RRRR & RRRB & RRRG & RRRM & \ldots \\ \ldots & BBBR & BBBB & BBBG & BBBM & \ldots \\ \ldots & GGGR & GGGB & GGGG & GGGM & \ldots \\ \ldots & MMMR & MMMB & MMMG & MMMM & \end{array}$$

Example: $\mathcal{S} = \{R, B, G, M\}$

There are $4 \cdot 3 = 12$ **lists without repetitions of length** 2

$$
\begin{array}{ccc}
RB & RG & RM \\
BR & BG & BM \\
GR & GB & GM \\
MR & MB & MG
\end{array}
$$

There are $4 \cdot 3 \cdot 2 = 24$ **lists without repetitions of length** 3

$$
\begin{array}{cccccc}
RBG & RBM & RGB & RGM & RMB & RMG \\
BRG & BRM & BGR & BGM & BMR & BMG \\
GRB & GRM & GBR & GBM & GMR & GMB \\
MRB & MRG & MBR & MBG & MGR & MGB
\end{array}
$$

Example: $\mathcal{S} = \{R, B, G, M\}$

$\mathcal{S}$ has 4 subsets of size 1

$$\{R\} \quad \{B\} \quad \{G\} \quad \{M\}$$

$\mathcal{S}$ has 6 subsets of size 2

$$\{RB\} \quad \{RG\} \quad \{RM\}$$
$$\{BG\} \quad \{BM\} \quad \{GM\}$$

$\mathcal{S}$ has 4 subsets of size 3

$$\{RBG\} \quad \{RBM\} \quad \{RGM\} \quad \{BGM\}$$

Permutations

Definition

- An n-permutation π is a 1-1 function from the set of numbers $\{1, 2, \ldots n\}$ to itself.
- $\pi(i) \neq \pi(j)$ for a permutation $\pi = (\pi(1), \pi(2), \ldots, \pi(n))$ for all $1 \leq i \neq j \leq n$.

Counting permutations

- There are $n! = n(n-1)(n-2)\cdots 2 \cdot 1$ permutations for $n \geq 1$.

Proof

- There are n options for $\pi(1)$.
- There are $n-1$ options for $\pi(2)$.
- $\vdots$
- There are 2 options for $\pi(n-1)$.
- There is only 1 option for $\pi(n)$.

Permutations: Examples

Small values

- $1! = 1$, $2! = 2$, $3! = 6$, $4! = 24$, $5! = 120$, $6! = 720$, $\ldots$

The 10 digits

- There are $10! = 3628800$ different ways to arrange the 10 digits $0, 1, \ldots, 9$.
- If numbers with less than 10 digits have leading zeros, then there are 10 billions ($10000000000 = 10^{10}$) numbers with 10 digits.
- Only 0.036288% of these numbers contain all the 10 digits.

The 26 letters

- There are $26! = 403291461126605635584000000$ arrangements of the 26 letters of the English alphabet.
- $26! \approx 4 * 10^{27}$ which is about 400 millions billions of billions.

There Are Exactly $10!$ Seconds in 6 Weeks

Preprocessing

- There are $60 = 3 \times 4 \times 5$ seconds in one minute.
- There are $60 = 2 \times \sqrt{9} \times 10$ minutes in one hour.
- There are $24 = \sqrt{9} \times 8$ hours in a day.
- There are 7 days in a week.
- There are 6 weeks.

$10!$ seconds

- The number S of seconds in 6 weeks is therefore:

$$\begin{aligned} S &= (3 \times 4 \times 5) \times (2 \times \sqrt{9} \times 10) \times (\sqrt{9} \times 8) \times 7 \times 6 \\ &= 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 \\ &= 10! \end{aligned}$$

Derangements

Definition

- A **derangement** is a permutation π on the numbers $1, 2, \ldots n$ such that $\pi(i) \neq i$ for all $1 \leq i \leq n$.
- A **derangement** is a permutation π on the numbers $1, 2, \ldots n$ without fixed points $(\neg\exists_{1 \leq i \leq n}(\pi(i) = i))$.

Examples

- (21) is the only derangement for $n = 2$.
- (231) and (312) are the only two derangements for $n = 3$ while the other four permutations (123), (132), (213), and (321) are not derangements because each contains at least one fixed point.
- The **9** derangements for $n = 4$ out of the 24 permutations:

(1234)	(1243)	(1324)	(1342)	(1423)	(1432)
(2134)	**(2143)**	(2314)	**(2341)**	**(2413)**	(2431)
(3124)	**(3142)**	(3214)	(3241)	**(3412)**	**(3421)**
(4123)	(4132)	(4213)	(4231)	**(4312)**	**(4321)**

Counting Derangements

Notation

- The number of derangements of n is $!n$ (the **subfactorial** of n).

Small n

- $!1 = 0$, $!2 = 1$, $!3 = 2$, $!4 = 9$, $!5 = 44$, $!6 = 265 \ldots$

Recursive formula

- $!1 = 0$, $!2 = 1$, and for $n \geq 3$:

$$!n = (n-1)(!(n-1)+!(n-2))$$

  * $!3 = 2(!2+!1) = 2(1+0) = 2$.
  * $!4 = 3(!3+!2) = 3(2+1) = 9$.
  * $!5 = 4(!4+!3) = 4(9+2) = 44$.
  * $!6 = 5(!5+!4) = 5(44+9) = 265$.

Derangements

Non-recursive formulas

- $!n = \left[\frac{n!}{e}\right] \approx \frac{n!}{2.718}$ where $[x]$ is the nearest integer to x.
- $!n = n! \sum_{i=0}^{n} \frac{-1^i}{i!} = n!(1/2 - 1/6 + 1/24 - 1/120 + \cdots)$.

Corollary

- About $1/e \approx 0.367879$ of the permutations are derangements.
- About 63% of the permutations have at least one fixed point.

An online resource

- `https://www.youtube.com/watch?v=pbXg5EI5t4c`

Lists With Repetitions

Definition

- $\mathcal{L} = (\ell_1, \ell_2, \ldots, \ell_k)$ is an ordered list of k objects from the set $\mathcal{S} = \{s_1, s_2, \ldots, s_n\}$ if $\ell_i \in \mathcal{S}$ for all $1 \leq i \leq k$.
- ℓ_i could be equal to ℓ_j for $1 \leq i < j \leq k$.

Counting the number of lists with repetitions

- There are n^k lists of length k from a set of size n.

Proof

- For each $1 \leq i \leq k$, there are n options for ℓ_i.
- For the k possible indices $1 \leq i \leq k$, there are n^k options for $(\ell_1, \ell_2, \ldots, \ell_k)$.

Lists With Repetitions: Examples

Numbers

- Assuming numbers have leading zeros, then there are 10 billions ($10000000000 = 10^{10}$) numbers (**lists**) with 10 digits.

Letters

- There are $26^3 = 17576$ possible three-letter words in English and $26^4 = 456976$ possible four-letter words in English.
- There are less than 200000 words in the Oxford English Dictionary!

Codes

- There are $10^4 = 10000$ possible codes for a 4-digit lock.
- The codes: 0000, 0001 . . . 4567 . . . 7766 . . . 9998, 9999.

Nesting Loops

Pseudocode

function $f(n)$ (* integer $n \geq 1$ *)
  $c = 0$
  for $i = 1$ **to** n **do**
    for $j = 1$ **to** n **do**
      for $k = 1$ **to** n **do**
        print (i, j, k)
        $c := c + 1$

Observations

- The function $f(n)$ prints all possible lists with repetitions (i, j, k) for which $i, j, k \in \{1, 2, \ldots, n\}$.
- The value of c at the end is n^3.

Strings

Definition

- An n-ary string of length k is an ordered list $\mathcal{D} = (d_1, d_2, \ldots, d_k)$ such that $d_i \in \{0, 1, \ldots, n-1\}$ for all $1 \le i \le k$.
- In a binary string $d_i = 0$ or $d_i = 1$ for all $1 \le i \le k$.

Counting strings

- There are n^k strings of length k.
- There are 2^k binary strings of length k.

Example: the 16 binary strings of length 4

0000	0001	0010	0011	0100	0101	0110	0111
1000	1001	1010	1011	1100	1101	1110	1111

Example: the 27 ternary strings of length 3

000	001	002	010	011	012	020	021	022
100	101	102	110	111	112	120	121	122
200	201	202	210	211	212	220	221	222

Non Homogeneous Lists

Definition

- In a non-homogeneous list, for $1 \leq i \leq k$, the entry ℓ_i gets its value from a different domain of objects denoted by $\mathcal{S}_i$.
- $\mathcal{L} = (\ell_1, \ell_2, \ldots, \ell_k)$ is an ordered non-homogeneous list of k objects if $\ell_i \in \mathcal{S}_i$ for all $1 \leq i \leq k$.

Counting the number of non-homogeneous lists

- Assume n_i is the size $\mathcal{S}_i$.
- Then there are $n_1 n_2 \cdots n_k$ non-homogeneous lists of length k.

Proof

- For each $1 \leq i \leq k$, there are n_i options for ℓ_i.
- For the k possible $1 \leq i \leq k$, there are $n_1 n_2 \cdots n_k$ options for $(\ell_1, \ell_2, \ldots, \ell_k)$.

Non Homogeneous Lists: Examples

Passwords

- There are $26^2 \cdot 10^4 = 6760000$ possible passwords of length 6 that must start with 2 letters and end with 4 digits.
- $\mathcal{S}_1 = \mathcal{S}_2 = \{A, B, \ldots, Z\}$ and $\mathcal{S}_3 = \mathcal{S}_4 = \mathcal{S}_5 = \mathcal{S}_6 = \{0, 1, \ldots, 9\}$.
- $AA0000$, $AA0001$, ... $CZ9999$, $DA0000$ ... $ZZ9998$, $ZZ9999$.

Taxi licenses

- There were only $10 \cdot 26 \cdot 10^2 = 26000$ possible taxi license numbers in New York city that must start with a digit followed by a letter and end with two digits.
- $\mathcal{S}_2 = \{A, B, \ldots, Z\}$ and $\mathcal{S}_1 = \mathcal{S}_3 = \mathcal{S}_4 = \{0, 1, \ldots, 9\}$.
- $0A00$, $0A01$ ... $5L99$, $5M00$ ... $9Z98$, $9Z99$.
- To add licenses, there are now licenses like $5L99_B$.

Nesting Loops

Pseudocode

function $f(r, s, t)$ (* integers $r, s, t \geq 1$ *)
  $c = 0$
  for $i = 1$ **to** r **do**
    for $j = 1$ **to** s **do**
      for $k = 1$ **to** t **do**
        print (i, j, k)
        $c := c + 1$

Observations

- The function $f(r, s, t)$ prints all possible non-homogeneous lists (i, j, k) for which $i \in \{1, 2, \ldots, r\}$, $j \in \{1, 2, \ldots, s\}$, and $k \in \{1, 2, \ldots, t\}$.
- The value of c at the end is $r \cdot s \cdot t$.

Lists Without Repetitions

Definition

- $\mathcal{L} = (\ell_1, \ell_2, \ldots, \ell_k)$ is an ordered list without repetitions of $k \leq n$ objects from the set $\mathcal{S} = \{s_1, s_2, \ldots, s_n\}$.
  - $\ell_i \in \mathcal{S}$ for all $1 \leq i \leq k$ and
  - $\ell_i \neq \ell_j$ for $1 \leq i < j \leq k$.

Counting the number of lists without repetitions

- There are $n^{\underline{k}} = n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!}$ lists without repetitions of length k on the numbers $1, 2, \ldots, n$.

Proof

- There are n options for ℓ_1.
- There are $n-1$ options for ℓ_2.

$$\vdots$$

- There are $n-k+1$ options for ℓ_k.
- In total there are $n(n-1)(n-2)\cdots(n-k+1)$ options.

Lists Without Repetitions: Remarks and Examples

Remarks

- $k \leq n$ because there are only n options for each ℓ_i.
- Permutations are lists without repetitions for which $k = n$.

Three-digit numbers

- There are $720 = 10 \cdot 9 \cdot 8$ three-digit numbers for which all the digits are different.
- 012, 013 . . . 309, 310 . . . 598, 601 . . . 986, 987.

Three-letter words

- There are $26 \cdot 25 \cdot 24 = 15600$ possible three-letter words in English with three different letters.
- The Scrabble Dictionary (OWL2) recognizes only 1015 three-letter words (words with or without repetitions).

Nesting Loops

Pseudocode

function $f(n)$ (* integer $n \geq 1$ *)
  $c = 0$
  for $i = 1$ **to** n **do**
    for $j = 1$ **to** n **if** $j \neq i$ **do**
      for $k = 1$ **to** n **if** $(k \neq i$ **and** $k \neq j)$ **do**
        print (i, j, k)
        $c := c + 1$

Observations

- The function $f(n)$ prints all possible lists without repetitions (i, j, k) for which $i, j, k \in \{1, 2, \ldots, n\}$ are three distinct numbers.
- The value of c at the end is $n(n-1)(n-2)$.

Subsets

Definition

- For $n \geq 1$ and $0 \leq k \leq n$, a set of n objects has $\binom{\mathbf{n}}{\mathbf{k}}$ different subsets of k objects.
- Equivalently, there are $\binom{\mathbf{n}}{\mathbf{k}}$ different ways to select k objects from a set of n objects.

Notations

- $\binom{n}{k}$ is called **"n choose k"**.
- Additional notations to $\binom{n}{k}$ are $C(n, k)$, $C_{n,k}$, C_n^k, and ${}^nC_k$.

Special cases

- $\binom{n}{0} = 1$ **for** $k = 0$**:** the empty set is the only subset with 0 objects and there is only one way to select 0 objects.
- $\binom{n}{n} = 1$ **for** $k = n$**:** the entire set is the only subset with n objects and there is only one way to select all the n objects.

Example of Subsets

All the subsets of the set $\mathcal{S} = \{R, B, G, M\}$

- There is only $\binom{4}{0} = 1$ way to choose zero colors from $\mathcal{S}$:
  - $\emptyset$
- There are $\binom{4}{1} = 4$ ways to choose one color from $\mathcal{S}$:
  - $\{R\}, \{B\}, \{G\}, \{M\}$
- There are $\binom{4}{2} = 6$ ways to choose two colors from $\mathcal{S}$:
  - $\{R, B\}, \{R, G\}, \{R, M\}, \{B, G\}, \{B, M\}, \{G, M\}$
- There are $\binom{4}{3} = 4$ ways to choose three colors from $\mathcal{S}$:
  - $\{R, B, G\}, \{R, B, M\}, \{R, G, M\}, \{B, G, M\}$
- There is only $\binom{4}{4} = 1$ way to choose four colors from $\mathcal{S}$:
  - $\{R, B, G, M\}$

A Formula for $\binom{n}{k}$

Theorem

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Equivalent formulas

$$\begin{aligned}
\binom{n}{k} &= \frac{n^{\underline{k}}}{k!} \\
&= \frac{n(n-1)(n-2)\cdots(n-k+1)}{k(k-1)(k-2)\cdots 2\cdot 1} \\
&= \frac{n}{1}\cdot\frac{(n-1)}{2}\cdot\frac{(n-2)}{3}\cdots\frac{(n-k+1)}{k} \\
&= \prod_{i=1}^{k}\left(\frac{n+1-i}{i}\right)
\end{aligned}$$

Proof of Theorem

$\binom{\mathbf{n}}{\mathbf{k}} = \frac{\mathbf{n}!}{\mathbf{k}!(\mathbf{n}-\mathbf{k})!}$

- There are n ways to select the first object, there are $n-1$ ways to select the second object, and so on ...
- There are $(n-k+1)$ ways to select the last k^{th} object.
- In total, there are

$$g(n,k) = n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!}$$

  ways to select an **ordered** list of k objects from a set of n objects.
- Each subset of k objects is selected in $k!$ different ordered lists and therefore, there are

$$\frac{g(n,k)}{k!} = \frac{n!}{k!(n-k)!}$$

  ways to select a subset of k objects from a set of n objects.

Nesting Loops

Pseudocode

function $f(n)$ (* integer $n \geq 1$ *)
  $c = 0$
  for $i = 1$ **to** $n - 2$ **do**
    for $j = i + 1$ **to** $n - 1$ **do**
      for $k = j + 1$ **to** n **do**
        print (i, j, k)
        $c := c + 1$

Observations

- The function $f(n)$ prints all possible subsets $\{i, j, k\}$ from the set $\{1, 2, \ldots, n\}$ such that $i < j < k$.
- The value of c at the end is $\binom{n}{3} = \frac{n!}{3!(n-3)!} = \frac{n(n-1)(n-2)}{6}$.

Calculating $\binom{n}{k}$ For Some Small Values of n and k

$n = 4$ and $k = 2$

$$\binom{4}{2} = \frac{4!}{2!2!} = \frac{24}{2 \cdot 2} = \frac{24}{4} = 6$$

$n = 5$ and $k = 3$

$$\binom{5}{3} = \frac{5!}{3!2!} = \frac{120}{6 \cdot 2} = \frac{120}{12} = 10$$

$n = 6$ and $k = 3$

$$\binom{6}{3} = \frac{6!}{3!3!} = \frac{720}{6 \cdot 6} = \frac{720}{36} = 20$$

Online calculator

https://www.omnicalculator.com/math/binomial-coefficient

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$ For Small k

$$\binom{n}{0} = \frac{n!}{0!n!} = 1$$

$$\binom{n}{1} = \frac{n!}{1!(n-1)!} = n$$

$$\binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{n(n-1)}{2}$$

$$\binom{n}{3} = \frac{n!}{3!(n-3)!} = \frac{n(n-1)(n-2)}{6}$$

$$\binom{n}{4} = \frac{n!}{4!(n-4)!} = \frac{n(n-1)(n-2)(n-3)}{24}$$

$$\binom{n}{5} = \frac{n!}{5!(n-5)!} = \frac{n(n-1)(n-2)(n-3)(n-4)}{120}$$

$\binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{n(n-1)}{2}$ For Small n

$$\binom{2}{2} = \frac{2!}{2!0!} = \frac{2}{2 \cdot 1} = 1 = \frac{2 \cdot 1}{2}$$

$$\binom{3}{2} = \frac{3!}{2!1!} = \frac{6}{2 \cdot 1} = 3 = \frac{3 \cdot 2}{2}$$

$$\binom{4}{2} = \frac{4!}{2!2!} = \frac{24}{2 \cdot 2} = 6 = \frac{4 \cdot 3}{2}$$

$$\binom{5}{2} = \frac{5!}{2!3!} = \frac{120}{2 \cdot 6} = 10 = \frac{5 \cdot 4}{2}$$

$$\binom{6}{2} = \frac{6!}{2!4!} = \frac{720}{2 \cdot 24} = 15 = \frac{6 \cdot 5}{2}$$

$$\binom{7}{2} = \frac{7!}{2!5!} = \frac{5040}{2 \cdot 120} = 21 = \frac{7 \cdot 6}{2}$$

$\binom{n}{3} = \frac{n!}{3!(n-3)!} = \frac{n(n-1)(n-2)}{6}$ For Small n

$$\binom{3}{3} = \frac{3!}{3!0!} = \frac{6}{6 \cdot 1} = 1 = \frac{3 \cdot 2 \cdot 1}{6}$$

$$\binom{4}{3} = \frac{4!}{3!1!} = \frac{24}{6 \cdot 1} = 4 = \frac{4 \cdot 3 \cdot 2}{6}$$

$$\binom{5}{3} = \frac{5!}{3!2!} = \frac{120}{6 \cdot 2} = 10 = \frac{5 \cdot 4 \cdot 3}{6}$$

$$\binom{6}{3} = \frac{6!}{3!3!} = \frac{720}{6 \cdot 6} = 20 = \frac{6 \cdot 5 \cdot 4}{6}$$

$$\binom{7}{3} = \frac{7!}{3!4!} = \frac{5040}{6 \cdot 24} = 35 = \frac{7 \cdot 6 \cdot 5}{6}$$

$$\binom{8}{3} = \frac{8!}{3!5!} = \frac{40320}{6 \cdot 120} = 56 = \frac{8 \cdot 7 \cdot 6}{6}$$

Symmetry

Theorem

$$\binom{n}{k} = \binom{n}{n-k}$$

Algebraic Proof

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!k!} = \binom{n}{n-k}$$

Symmetry

Theorem

$$\binom{n}{k} = \binom{n}{n-k}$$

Combinatorial Proof

- Selecting a subset of k objects from a set of n objects is **equivalent** to selecting the complement subset of the $n-k$ objects not in the set.
- Therefore, the number of subsets of size k is equal to the number of subsets of size $n-k$.

Example: $\mathcal{S} = \{C, R, B, G, M\}$

Matching the $\binom{5}{2} = 10$ two-subsets with the $\binom{5}{3} = 10$ three-subsets

$$\{G, M\} \longleftrightarrow \{C, R, B\}$$
$$\{B, M\} \longleftrightarrow \{C, R, G\}$$
$$\{B, G\} \longleftrightarrow \{C, R, M\}$$
$$\{R, M\} \longleftrightarrow \{C, B, G\}$$
$$\{R, G\} \longleftrightarrow \{C, B, M\}$$
$$\{R, B\} \longleftrightarrow \{C, G, M\}$$
$$\{C, M\} \longleftrightarrow \{R, B, G\}$$
$$\{C, G\} \longleftrightarrow \{R, B, M\}$$
$$\{C, B\} \longleftrightarrow \{R, G, M\}$$
$$\{C, R\} \longleftrightarrow \{B, G, M\}$$

Subsets as Binary Strings

Theorem

- The number of subsets of a set S with n objects is the same as the number of binary strings of length n which is 2^n.

Proof

- A subset $R \subseteq S$ can be represented by the binary string $(b_1, b_2, \ldots, b_n)$ in which $b_i = 1$ if $s_i \in R$ and $b_i = 0$ if $s_i \notin R$.
- A binary string $(b_1, b_2, \ldots, b_n)$ can be represented by a subset $R \subseteq S$ such that $s_i \in R$ if $b_i = 1$ and $s_i \notin R$ if $b_i = 0$.
- Thus, there is a **one-to-one mapping** from the set 2^S of all the subsets of S to the set of all binary strings of length n.
- Therefore, $|2^S| = 2^n$.

Example: The $2^4 = 16$ Subsets of $\{R, B, G, M\}$

$$
\begin{aligned}
\emptyset &\equiv (0,0,0,0) \\
\{R\} &\equiv (1,0,0,0) \\
\{B\} &\equiv (0,1,0,0) \\
\{G\} &\equiv (0,0,1,0) \\
\{M\} &\equiv (0,0,0,1) \\
\{R,B\} &\equiv (1,1,0,0) \\
\{R,G\} &\equiv (1,0,1,0) \\
\{R,M\} &\equiv (1,0,0,1)
\end{aligned}
$$

$$
\begin{aligned}
\{B,G\} &\equiv (0,1,1,0) \\
\{B,M\} &\equiv (0,1,0,1) \\
\{G,M\} &\equiv (0,0,1,1) \\
\{R,B,G\} &\equiv (1,1,1,0) \\
\{R,B,M\} &\equiv (1,1,0,1) \\
\{R,G,M\} &\equiv (1,0,1,1) \\
\{B,G,M\} &\equiv (0,1,1,1) \\
\{R,B,G,M\} &\equiv (1,1,1,1)
\end{aligned}
$$

Subsets as Binary Strings

Corollary

- For $0 \leq k \leq n$, there are $\binom{n}{k}$ binary strings of length n with exactly k ones.

Proof

- By definition, a set of size n has $\binom{n}{k}$ subsets of size k.
- The one-to-one mapping in the proof of the theorem maps all the sets of size k to all the binary strings with exactly k ones.

Special cases

- The null set $\emptyset$ is equivalent to the all-0 binary string $(0, 0, \ldots, 0)$.
- The set itself is equivalent to the all-1 binary string $(1, 1, \ldots, 1)$.
- A binary string with a singleton 1 is equivalent to a singleton subset $\{x\}$ that contains one of the objects x from the set.

Recursive Formula

Recursion

$$\text{for all integers } n \geq 0 \qquad \binom{n}{0} = \binom{n}{n} = 1$$

$$\text{for all integers } 1 \leq k \leq n-1 \qquad \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Examples

- $6 = \binom{4}{2} = \binom{3}{1} + \binom{3}{2} = 3 + 3 = 6$
- $10 = \binom{5}{3} = \binom{4}{2} + \binom{4}{3} = 6 + 4 = 10$
- $20 = \binom{6}{3} = \binom{5}{2} + \binom{5}{3} = 10 + 10 = 20$
- $15 = \binom{6}{4} = \binom{5}{3} + \binom{5}{4} = 10 + 5 = 15$
- $35 = \binom{7}{4} = \binom{6}{3} + \binom{6}{4} = 20 + 15 = 35$

Recursive Formula: Combinatorial Proof I

$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$: Informal proof by induction

.

- Consider a set with n objects $\mathcal{S} = \{s_1, s_2, \ldots, s_n\}$.
- There are two options for selecting a subset $\mathcal{R}$ of $\mathcal{S}$ with k objects:
  * $s_n \in \mathcal{R}$: There are $\binom{n-1}{k-1}$ different ways to select additional $k-1$ objects out of $s_1, s_2, \ldots, s_{n-1}$.
  * $s_n \notin \mathcal{R}$: There are $\binom{n-1}{k}$ different ways to select k objects out of $s_1, s_2, \ldots, s_{n-1}$.
- In total the number of ways to select k objects from a set of n objects $\binom{n}{k}$ is also

$$\binom{n-1}{k-1} + \binom{n-1}{k}$$

Subsets of $\mathcal{S} = \{R, B, G, M\}$

The six subsets of size 2 from the set $\mathcal{S} = \{R, B, G, M\}$

- There are $\binom{4}{2} = 6$ ways to choose two colors from S:
  - $\{R, B\}, \{R, G\}, \{R, M\}, \{B, G\}, \{B, M\}, \{G, M\}$
- There are $\binom{3}{1} = 3$ ways to choose two colors from S where one of them is Magenta:
  - $\{R, M\}, \{B, M\}, \{G, M\}$
- There are $\binom{3}{2} = 3$ ways to choose two colors from S none of them is Magenta:
  - $\{R, B\}, \{R, G\}, \{B, G\}$

Subsets of $\mathcal{S} = \{C, R, B, G, M\}$

The ten subsets of size 3 from the set $\mathcal{S} = \{C, R, B, G, M\}$

- There are $\binom{5}{3} = 10$ ways to choose three colors from S:
  - $\{C, R, B\}, \{C, R, G\}, \{C, R, M\}, \{C, B, G\}, \{C, B, M\}, \{C, G, M\}$
  - $\{R, B, G\}, \{R, B, M\}, \{R, G, M\}, \{B, G, M\}$
- There are $\binom{4}{2} = 6$ ways to choose three colors from S where one of them is Magenta:
  - $\{C, R, M\}, \{C, B, M\}, \{C, G, M\}, \{R, B, M\}, \{R, G, M\}, \{B, G, M\}$
- There are $\binom{4}{3} = 4$ ways to choose three colors from S none of them is Magenta:
  - $\{C, R, B\}, \{C, R, G\}, \{C, B, G\}, \{R, B, G\}$

Recursive Formula: Algebraic Proof

$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

$$
\begin{aligned}
\binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!} \\
&= \frac{k(n-1)!}{k(k-1)!(n-k)!} + \frac{(n-k)(n-1)!}{(n-k)k!(n-k-1)!} \\
&= \frac{k(n-1)!}{k!(n-k)!} + \frac{(n-k)(n-1)!}{k!(n-k)!} \\
&= \frac{k(n-1)! + (n-k)(n-1)!}{k!(n-k)!} \\
&= \frac{(k+(n-k))(n-1)!}{k!(n-k)!} \\
&= \frac{n(n-1)!}{k!(n-k)!} \\
&= \frac{n!}{k!(n-k)!} \\
&= \binom{n}{k}
\end{aligned}
$$

Recursive Formula: Combinatorial Proof II

$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$: Informal proof by induction

- $\binom{n}{k}$ is the number of binary strings of length n containing k 1's.
- Some strings start with a 1 while the rest start with a 0.
- Binary strings which start with a 1:
  - After the 1, out of the remaining $n-1$ bits $k-1$ bits must be 1.
  - There are $\binom{n-1}{k-1}$ such binary strings.
- Binary strings which start with a 0:
  - After the 0, out of the remaining $n-1$ bits k must be 1.
  - There are $\binom{n-1}{k}$ such binary strings.
- Thus, the number of binary strings of length n containing k 1's is

$$\binom{n-1}{k-1} + \binom{n-1}{k}$$

Example

The 10 binary strings of length 5 with exactly 2 ones

$$(11000)\ \ (10100)\ \ (10010)\ \ (10001)\ \ (01100)$$

$$(01010)\ \ (01001)\ \ (00110)\ \ (00101)\ \ (00011)$$

- There are $\binom{4}{1} = 4$ strings that start with 1:

$$(11000)\ \ (10100)\ \ (10010)\ \ (10001)$$

- There are $\binom{4}{2} = 6$ strings that start with 0:

$$(01100)\ \ (01010)\ \ (01001)\ \ (00110)\ \ (00101)\ \ (00011)$$

- The total number of strings is $\binom{5}{2} = \binom{4}{1} + \binom{4}{2} = 4 + 6 = 10$

Example

The 20 binary strings of length 6 with exactly 3 zeros

(000111) (001011) (001101) (001110) (010011)

(010101) (010110) (011001) (011010) (011100)

(100011) (100101) (100110) (101001) (101010)

(101100) (110001) (110010) (110100) (111000)

- There are $\binom{5}{2} = 10$ strings that start with 0:

  (000111) (001011) (001101) (001110) (010011)

  (010101) (010110) (011001) (011010) (011100)

- There are $\binom{5}{3} = 10$ strings that start with 1:

  (100011) (100101) (100110) (101001) (101010)

  (101100) (110001) (110010) (110100) (111000)

- The total number of strings is $\binom{6}{3} = \binom{5}{2} + \binom{5}{3} = 10 + 10 = 20$.

2^{nd} Recursive Formula

Theorem

- For $1 \leq k \leq n$,

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$$

Proof

$$\begin{aligned}
\binom{n}{k} &= \frac{n!}{k!(n-k)!} \\
&= \frac{n(n-1)!}{k(k-1)!(n-k)!} \\
&= \frac{n}{k} \cdot \frac{(n-1)!}{(k-1)!(n-k)!} \\
&= \frac{n}{k}\binom{n-1}{k-1}
\end{aligned}$$

3^{rd} Recursive Formula

Theorem

- For $1 \le k \le n$,

$$\binom{n}{k} = \frac{n}{n-k}\binom{n-1}{k}$$

Proof

$$\begin{aligned}\binom{n}{k} &= \frac{n!}{k!(n-k)!} \\ &= \frac{n(n-1)!}{k!(n-k)(n-k-1)!} \\ &= \frac{n}{n-k} \cdot \frac{(n-1)!}{k!(n-k-1)!} \\ &= \frac{n}{n-k}\binom{n-1}{k}\end{aligned}$$

4^{th} Recursive Formula

Theorem

- For $1 \le k \le n$,

$$\binom{n}{k} = \frac{n+1-k}{k}\binom{n}{k-1}$$

Proof

$$\begin{aligned}
\binom{n}{k} &= \frac{n!}{k!(n-k)!} \\
&= \frac{(n+1-k)n!}{k(k-1)!(n+1-k)(n-k)!} \\
&= \frac{n+1-k}{k} \cdot \frac{n!}{(k-1)!(n+1-k)!} \\
&= \frac{n+1-k}{k}\binom{n}{k-1}
\end{aligned}$$

Examples

$\binom{7}{3} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35$

- $\binom{7}{3} = \binom{6}{2} + \binom{6}{3} = 15 + 20 = 35$
- $\binom{7}{3} = \frac{7}{3} \cdot \binom{6}{2} = \frac{7}{3} \cdot \frac{6 \cdot 5}{2 \cdot 1} = \frac{7}{3} \cdot 15 = 35$
- $\binom{7}{3} = \frac{7}{4} \cdot \binom{6}{3} = \frac{7}{4} \cdot \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} = \frac{7}{4} \cdot 20 = 35$
- $\binom{7}{3} = \frac{5}{3} \cdot \binom{7}{2} = \frac{5}{3} \cdot \frac{7 \cdot 6}{2 \cdot 1} = \frac{5}{3} \cdot 21 = 35$

$\binom{8}{3} = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} = 56$

- $\binom{8}{3} = \binom{7}{2} + \binom{7}{3} = 21 + 35 = 56$
- $\binom{8}{3} = \frac{8}{3} \cdot \binom{7}{2} = \frac{8}{3} \cdot \frac{7 \cdot 6}{2 \cdot 1} = \frac{8}{3} \cdot 21 = 56$
- $\binom{8}{3} = \frac{8}{5} \cdot \binom{7}{3} = \frac{8}{5} \cdot \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = \frac{8}{5} \cdot 35 = 56$
- $\binom{8}{3} = \frac{6}{3} \cdot \binom{8}{2} = 2 \cdot \frac{8 \cdot 7}{2 \cdot 1} = 2 \cdot 28 = 56$

Another Proof for the Main Recursive Formula

$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

- The 2nd recursive formula implies that

$$\binom{n-1}{k-1} = \frac{k}{n}\binom{n}{k}$$

- The 3rd recursive formula implies that

$$\binom{n-1}{k} = \frac{n-k}{n}\binom{n}{k}$$

- Therefore,

$$\begin{aligned}\binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{k}{n}\binom{n}{k} + \frac{n-k}{n}\binom{n}{k} \\ &= \frac{(k+n-k)}{n}\binom{n}{k} \\ &= \binom{n}{k}\end{aligned}$$

$(x+y)^n$

Theorem

$$
\begin{aligned}
(x+y)^n &= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \\
&= \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y^1 + \cdots + \binom{n}{k} x^{n-k} y^k + \cdots + \binom{n}{n-1} x^1 y^{n-1} + \binom{n}{n} x^0 y^n \\
&= x^n + n x^{n-1} y + \frac{n(n-1)}{2} x^{n-2} y^2 + \cdots + \frac{n(n-1)}{2} x^2 y^{n-2} + n x y^{n-1} + y^n
\end{aligned}
$$

The binomial coefficients

- Based on the above theorem, $\binom{n}{k}$ is called **a binomial coefficient**.

$(x+y)^n$

Proof

- By definition,

$$(x+y)^n = \underbrace{(x+y)\cdot(x+y)\cdots(x+y)\cdots(x+y)\cdot(x+y)}_{n}$$

- Using the distributive laws to get the product $x^{n-k}y^k$:
  - $*$ select k of the n terms to contribute a y to the product.
  - $*$ select the other $n-k$ terms to contribute an x to the product.
- The coefficient of $x^{n-k}y^k$ is therefore $\binom{n}{k}$:
  - $*$ the number of ways to select k objects from a set of size n.
- Summing over all possible values of k from 0 to n implies that

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$(x+y)^n$

Small n

$$\begin{aligned}(x+y)^1 &= \binom{1}{0}x^1y^0 + \binom{1}{1}x^0y^1 \\ &= x+y\end{aligned}$$

$$\begin{aligned}(x+y)^2 &= \binom{2}{0}x^2y^0 + \binom{2}{1}x^1y^1 + \binom{2}{2}x^0y^2 \\ &= x^2+2xy+y^2\end{aligned}$$

$$\begin{aligned}(x+y)^3 &= \binom{3}{0}x^3y^0 + \binom{3}{1}x^2y^1 + \binom{3}{2}x^1y^2 + \binom{3}{3}x^0y^3 \\ &= x^3+3x^2y+3xy^2+y^3\end{aligned}$$

$$\begin{aligned}(x+y)^4 &= \binom{4}{0}x^4y^0 + \binom{4}{1}x^3y^1 + \binom{4}{2}x^2y^2 + \binom{4}{3}x^1y^3 + \binom{4}{4}x^0y^4 \\ &= x^4+4x^3y+6x^2y^2+4xy^3+y^4\end{aligned}$$

Example

$$\begin{aligned}(3+2)^4 &= 3^4 + 4\cdot 3^3\cdot 2 + 6\cdot 3^2\cdot 2^2 + 4\cdot 3\cdot 2^3 + 2^4 \\ &= 81+216+216+96+16 = 625 = 5^4\end{aligned}$$

$(1+y)^n$

Corollary

$$\begin{aligned}(1+y)^n &= \binom{n}{0}y^0+\binom{n}{1}y^1+\binom{n}{2}y^2+\cdots+\binom{n}{n-1}y^{n-1}+\binom{n}{n}y^n \\ &= \sum_{k=0}^{n}\binom{n}{k}y^k\end{aligned}$$

Example

$$\begin{aligned}(1+y)^4 &= \binom{4}{0}y^0+\binom{4}{1}y^1+\binom{4}{2}y^2+\binom{4}{3}y^3+\binom{4}{4}y^4 \\ &= 1+4y+6y^2+4y^3+y^4\end{aligned}$$

$$\begin{aligned}(1+4)^4 &= 1+4\cdot 4+6\cdot 4^2+4\cdot 4^3+4^4 \\ &= 1+16+96+256+256=625=5^4\end{aligned}$$

$(x+1)^n$

Corollary

$$
\begin{aligned}
(x+1)^n &= \binom{n}{0}x^n + \binom{n}{1}x^{n-1} + \binom{n}{2}x^{n-2} + \cdots + \binom{n}{n-1}x^1 + \binom{n}{n}x^0 \\
&= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} \\
&= \sum_{k=0}^{n} \binom{n}{n-k} x^{n-k} \\
&= \sum_{\ell=0}^{n} \binom{n}{\ell} x^{\ell}
\end{aligned}
$$

Example

$$
\begin{aligned}
(x+1)^4 &= \binom{4}{0}x^4 + \binom{4}{1}x^3 + \binom{4}{2}x^2 + \binom{4}{1}x^1 + \binom{4}{4}x^0 \\
&= x^4 + 4x^3 + 6x^2 + 4x + 1
\end{aligned}
$$

Sum of All Binomial Coefficients for a Given n

Examples

$$
\begin{aligned}
\binom{0}{0} &= 1 = 2^0 \\
\binom{1}{0} + \binom{1}{1} &= 1 + 1 = 2 = 2^1 \\
\binom{2}{0} + \binom{2}{1} + \binom{2}{2} &= 1 + 2 + 1 = 4 = 2^2 \\
\binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} &= 1 + 3 + 3 + 1 = 8 = 2^3 \\
\binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4} &= 1 + 4 + 6 + 4 + 1 = 16 = 2^4
\end{aligned}
$$

Sum of All Binomial Coefficients for a Given n

Theorem

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Proof

$$\begin{aligned}
2^n &= (1+1)^n \\
&= \binom{n}{0} 1^n 1^0 + \binom{n}{1} 1^{n-1} 1^1 + \binom{n}{2} 1^{n-2} 1^2 + \cdots + \binom{n}{n} 1^0 1^n \\
&= \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} \\
&= \sum_{k=0}^{n} \binom{n}{k}
\end{aligned}$$

Number of Subsets

Theorem

- A set of size n has 2^n subsets.

Proof

- By definition, a set of size n has $\binom{n}{k}$ subsets of size k for $0 \leq k \leq n$.
- Therefore, the number of subsets of a set of size n is

$$\sum_{k=0}^{k=n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}$$

- By the previous theorem this sum equals 2^n.

Computing 3^n with Powers of 2

Corollary

$$
\begin{aligned}
3^n &= (1+2)^n \\
&= \binom{n}{0} 1^n 2^0 + \binom{n}{1} 1^{n-1} 2^1 + \binom{n}{2} 1^{n-2} 2^2 + \cdots + \binom{n}{n} 1^0 2^n \\
&= \binom{n}{0} 2^0 + \binom{n}{1} 2^1 + \binom{n}{2} 2^2 + \cdots + \binom{n}{n} 2^n \\
&= \sum_{k=0}^{n} \binom{n}{k} 2^k
\end{aligned}
$$

Computing 3^n with Powers of 2

$n = 3$

$$27 = 3^3 = \binom{3}{0}2^0 + \binom{3}{1}2^1 + \binom{3}{2}2^2 + \binom{3}{3}2^3$$
$$= 1 \cdot 1 + 3 \cdot 2 + 3 \cdot 4 + 1 \cdot 8$$
$$= 1 + 6 + 12 + 8$$

$n = 4$

$$81 = 3^4 = \binom{4}{0}2^0 + \binom{4}{1}2^1 + \binom{4}{2}2^2 + \binom{4}{3}2^3 + \binom{4}{4}2^4$$
$$= 1 \cdot 1 + 4 \cdot 2 + 6 \cdot 4 + 4 \cdot 8 + 1 \cdot 16$$
$$= 1 + 8 + 24 + 32 + 16$$

Solving Problems

Counting rectangles in a square grid

- Animation:
  https://www.youtube.com/watch?v=GfODdLHwWZw

Counting paths in a rectangular grid

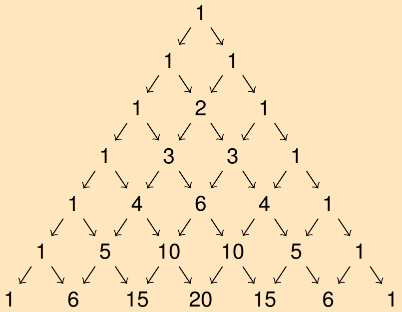
- Lecture:
  https://www.youtube.com/watch?v=fpnNaAU0iPk&list=PLmdFyQYShrjfPLdHQxuNWvh2ct666Na3z
- Animation:
  https://www.youtube.com/watch?v=9YUl0k2FYzc

Pascal's Triangle

Definition

- A **triangular** array of positive integers.
- The left edge and the right edge are all 1.
- Construct the rows from top to bottom.
- Each number is the sum of the two numbers above it diagonally.

The first 7 rows



Pascal's Triangle

The number triangle vs. the binomial coefficient triangle

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}
$$

$$
\begin{array}{c}
\binom{0}{0} \\
\binom{1}{0} \; \binom{1}{1} \\
\binom{2}{0} \; \binom{2}{1} \; \binom{2}{2} \\
\binom{3}{0} \; \binom{3}{1} \; \binom{3}{2} \; \binom{3}{3} \\
\binom{4}{0} \; \binom{4}{1} \; \binom{4}{2} \; \binom{4}{3} \; \binom{4}{4} \\
\binom{5}{0} \; \binom{5}{1} \; \binom{5}{2} \; \binom{5}{3} \; \binom{5}{4} \; \binom{5}{5} \\
\binom{6}{0} \; \binom{6}{1} \; \binom{6}{2} \; \binom{6}{3} \; \binom{6}{4} \; \binom{6}{5} \; \binom{6}{6} \\
\binom{7}{0} \; \binom{7}{1} \; \binom{7}{2} \; \binom{7}{3} \; \binom{7}{4} \; \binom{7}{5} \; \binom{7}{6} \; \binom{7}{7}
\end{array}
$$

Pascal's Triangle

The first 13 rows

												1												
											1		1											
										1		2		1										
									1		3		3		1									
								1		4		6		4		1								
							1		5		10		10		5		1							
						1		6		15		20		15		6		1						
					1		7		21		35		35		21		7		1					
				1		8		28		56		70		56		28		8		1				
			1		9		36		84		126		126		84		36		9		1			
		1		10		45		120		210		252		210		120		45		10		1		
	1		11		55		165		330		462		462		330		165		55		11		1	
1		12		66		220		495		792		924		792		495		220		66		12		1

Pascal's Triangle

The sum of the first 9 rows

$$1 = \mathbf{1} = \mathbf{2^0}$$
$$1 + 1 = \mathbf{2} = \mathbf{2^1}$$
$$1 + 2 + 1 = \mathbf{4} = \mathbf{2^2}$$
$$1 + 3 + 3 + 1 = \mathbf{8} = \mathbf{2^3}$$
$$1 + 4 + 6 + 4 + 1 = \mathbf{16} = \mathbf{2^4}$$
$$1 + 5 + 10 + 10 + 5 + 1 = \mathbf{32} = \mathbf{2^5}$$
$$1 + 6 + 15 + 20 + 15 + 6 + 1 = \mathbf{64} = \mathbf{2^6}$$
$$1 + 7 + 21 + 35 + 35 + 21 + 7 + 1 = \mathbf{128} = \mathbf{2^7}$$
$$1 + 8 + 28 + 56 + 70 + 56 + 28 + 8 + 1 = \mathbf{256} = \mathbf{2^8}$$

Pascal's Triangle

Online video resources

- The mathematical secrets of Pascal's triangle
  https://www.youtube.com/watch?v=XMriWTvPXHI
- What You Don't Know About Pascal's Triangle?
  https://www.youtube.com/watch?v=J0I1NuxUcpQ
- Pascal's Triangle - Numberphile
  https://www.youtube.com/watch?v=0iMtlus-afo

Online text resources

- Summary of facts
  https://www.mathsisfun.com/pascals-triangle.html
- Mysterious Patterns in Pascal's Triangle
  https://www.cantorsparadise.com/mysterious-patterns-in-pascals-triangle-b8bad8d494e3

Triangular Numbers

Definition

- The triangular number T_n counts objects arranged in an equilateral triangle whose sides each has n objects.

Illustrations

- https://cdn1.byjus.com/wp-content/uploads/2016/06/triangular-numbers.jpg

Animation

- https://www.youtube.com/watch?v=TgQn8snKGtw

Recursive definition

$$T_n = \begin{cases} 1 & \text{for } n = 1 \\ T_{n-1} + n & \text{for } n > 1 \end{cases}$$

Triangular Numbers

Closed-form expression

$$T_n = n + (n-1) + (n-2) + \cdots + 2 + 1 = \frac{n(n+1)}{2} = \binom{n+1}{2}$$

Proof

- By induction

Visual proof for: $\mathbf{n + (n-1) + \cdots + 1 = \binom{n+1}{2}}$

- `https://www.youtube.com/watch?v=r5WloGGwVUg`

Triangular Numbers in Pascal's Triangle

1

1 1

1 2 **1**

1 **3** **3** 1

1 4 **6** 4 1

1 5 **10** **10** 5 1

1 6 **15** 20 **15** 6 1

1 7 **21** 35 35 **21** 7 1

1 8 **28** 56 70 56 **28** 8 1

1 9 **36** 84 126 126 84 **36** 9 1

1 10 **45** 120 210 252 210 120 **45** 10 1

1 11 **55** 165 330 462 462 330 165 **55** 11 1

Sum of Two Consecutive Triangular Numbers

Identity for integer n $\geq$ 2

$$T_{n-1} + T_n = \binom{n}{2} + \binom{n+1}{2} = n^2$$

Small values of n

- $\binom{2}{2} + \binom{3}{2} = 1 + 3 = 4 = 2^2$
- $\binom{3}{2} + \binom{4}{2} = 3 + 6 = 9 = 3^2$
- $\binom{4}{2} + \binom{5}{2} = 6 + 10 = 16 = 4^2$
- $\binom{5}{2} + \binom{6}{2} = 10 + 15 = 25 = 5^2$
- $\binom{6}{2} + \binom{7}{2} = 15 + 21 = 36 = 6^2$

Demo

- https://www.youtube.com/watch?v=ONCTEp-v1hU

Sum of Two Consecutive Triangular Numbers

Identity for integer n $\geq$ 2

$$T_{n-1} + T_n = \binom{n}{2} + \binom{n+1}{2} = n^2$$

Proof

$$\begin{aligned}
\binom{n}{2} + \binom{n+1}{2} &= \frac{n(n-1)}{2} + \frac{(n+1)n}{2} \\
&= \frac{n(n-1) + (n+1)n}{2} \\
&= \frac{n((n-1) + (n+1))}{2} \\
&= \frac{n \cdot 2n}{2} \\
&= n^2
\end{aligned}$$

Alternating Sum of First 2n Triangular Numbers

Identity for integer n ≥ 1

$$\sum_{k=1}^{2n}(-1)^k T_k = T_{2n} - T_{2n-1} + T_{2n-2} - T_{2n-3} + \cdots + T_2 - T_1 = 2T_n$$

Small values of n

- $3 - 1 = 2 = 2T_1$
- $10 - 6 + 3 - 1 = 6 = 2T_2$
- $21 - 15 + 10 - 6 + 3 - 1 = 12 = 2T_3$
- $36 - 28 + 21 - 15 + 10 - 6 + 3 - 1 = 20 = 2T_4$

Visual proof

- https://www.youtube.com/watch?v=Z_FNVP5eJrI

Alternating Sum of First 2n Triangular Numbers

Identity for integer n ≥ 1

$$T_{2n} - T_{2n-1} + T_{2n-2} - T_{2n-3} + \cdots + T_2 - T_1 = 2T_n$$

Proof

$$\begin{aligned}
(T_{2n} - T_{2n-1}) + (T_{2n-2} - T_{2n-3}) + \cdots + (T_2 - T_1) &= 2n + (2n-2) + \cdots + 2 \\
&= 2(n + (n-1) + \cdots + 1) \\
&= 2T_n \\
&= 2 \cdot \frac{n(n+1)}{2} \\
&= n(n+1)
\end{aligned}$$

Alternating Sum of First 2n − 1 Triangular Numbers

Identity for integer n ≥ 1

$$\sum_{k=1}^{2n-1} (-1)^k T_k = T_{2n-1} - T_{2n-2} + T_{2n-3} - T_{2n-4} + \cdots + T_3 - T_2 + T_1 = n^2$$

Small values of n

- $1 = 1 = 1^2$
- $6 - 3 + 1 = 4 = 2^2$
- $15 - 10 + 6 - 3 + 1 = 9 = 3^2$
- $28 - 21 + 15 - 10 + 6 - 3 + 1 = 16 = 4^2$

Visual proof

- `https://www.youtube.com/watch?v=dRa3ItqEZwM`

Alternating Sum of First 2n − 1 Triangular Numbers

Identity for integer n $\geq$ 1

$$T_{2n-1} - T_{2n-2} + T_{2n-3} - T_{2n-4} + \cdots + T_3 - T_2 + T_1 = n^2$$

Proof

$$\begin{aligned}
(T_{2n-1} - T_{2n-2}) + (T_{2n-3} - T_{2n-2}) + \cdots + (T_3 - T_2) + T_1 &= (2n-1) + (2n-3) + \cdots + 3 + 1 \\
&= ((2n-2)+1) + ((2n-4)+1) + (2+1) + 1 \\
&= ((2n-2) + (2n-4) + \cdots + 2) + n \\
&= 2((n-1) + (n-2) + \cdots + 1) + n \\
&= 2 \cdot \frac{(n-1)n}{2} + n \\
&= (n-1)n + n \\
&= n^2 - n + n \\
&= n^2
\end{aligned}$$

A Triangular Number Recurrence

Identity for integer n $\geq$ 1

$$T_{n+1} = \frac{n+2}{n} T_n$$

Proof sketch

$$nT_{n+1} = n\frac{(n+1)(n+2)}{2} = (n+2)\frac{n(n+1)}{2} = (n+2)T_n$$

Examples

- $\frac{3+2}{3} \cdot T_3 = \frac{5}{3} \cdot \; 6 = \frac{5 \cdot 6}{3} = 10 = T_4$
- $\frac{4+2}{4} \cdot T_4 = \frac{6}{4} \cdot 10 = \frac{6 \cdot 10}{4} = 15 = T_5$

Visual Proof

- https://www.youtube.com/watch?v=69TvfNA0Lxc

Sum of First n Cubes

Identity for integer n $\geq$ 1

$$\begin{aligned}\sum_{i=1}^{n} i^3 &= 1+8+27+\cdots+(n-1)^3+n^3 \\ &= T_n^2 \\ &= \binom{n+1}{2}^2 \\ &= \left(\frac{n(n+1)}{2}\right)^2 \\ &= (1+2+3+\cdots+(n-1)+n)^2\end{aligned}$$

Visual Proofs

- Figure: `https://i.sstatic.net/XHc4q.png`
- Animation 1: `https://www.youtube.com/watch?v=YQLicI8R4Gs`
- Animation 2: `https://www.youtube.com/watch?v=Ye9OPNqV9FA`
- Animation 3: `https://www.youtube.com/watch?v=NxOcT_VKQR0`
- Animation 4: `https://www.youtube.com/watch?v=jWpyrXYZNiI`
- Animation 5: `https://www.youtube.com/watch?v=d1yM6Rq7Tfw`

Sum of First n Cubes

Examples with Small n

$$
\begin{aligned}
1 &= 1 = 1^2 = T_1^2 \\
1+8 &= 9 = 3^2 = T_2^2 \\
1+8+27 &= 36 = 6^2 = T_3^2 \\
1+8+27+64 &= 100 = 10^2 = T_4^2 \\
1+8+27+64+125 &= 225 = 15^2 = T_5^2 \\
1+8+27+64+125+216 &= 441 = 21^2 = T_6^2 \\
1+8+27+64+125+216+343 &= 784 = 28^2 = T_7^2
\end{aligned}
$$

Sum of First n Cubes

Identity for integer n $\geq$ 1

$$\sum_{i=1}^{n} i^3 = T_n^2$$

Proof sketch

- The proof is based on the following identity for i^3

$$T_i^2 - T_{i-1}^2 = (T_i + T_{i-1})(T_i - T_{i-1}) = i^2 \cdot i = i^3$$

- As a result

$$\begin{aligned}\sum_{i=1}^{n} i^3 &= \sum_{i=1}^{n} (T_i^2 - T_{i-1}^2) \\ &= (T_n^2 - T_{n-1}^2) + (T_{n-1}^2 - T_{n-2}^2) + \cdots + (T_1^2 - T_0^2) \\ &= T_n^2\end{aligned}$$

The Triangular Number of a Sum

Identity for integers $n \geq 1$ and $k \geq 1$

$$T_{n+k} = T_n + T_k + n \cdot k$$

Examples

- $T_7 = T_3 + T_4 + 3 \cdot 4 = \ 6 + 10 + 12 = 28$
- $T_9 = T_2 + T_7 + 2 \cdot 7 = \ 3 + 28 + 14 = 45$

Proof

$$\begin{aligned} T_{n+k} &= \frac{(n+k)(n+k+1)}{2} \\ &= \frac{n(n+1) + k(k+1) + n \cdot k + k \cdot n}{2} \\ &= \frac{n(n+1)}{2} + \frac{k(k+1)}{2} + \frac{n \cdot k + k \cdot n}{2} \\ &= T_n + T_k + n \cdot k \end{aligned}$$

The Triangular Number of a Product

Identity for integers $n > 1$ and $k > 1$

$$T_{n \cdot k} = T_n \cdot T_k + T_{n-1} \cdot T_{k-1}$$

Examples

- $T_8 = T_2 \cdot T_4 + T_1 \cdot T_3 = 3 \cdot 10 + 1 \cdot 6 = 30 + 6 = 36$
- $T_9 = T_3 \cdot T_3 + T_2 \cdot T_2 = 6 \cdot 6 + 3 \cdot 3 = 36 + 9 = 45$
- $T_{12} = T_4 \cdot T_3 + T_3 \cdot T_2 = 10 \cdot 6 + 6 \cdot 3 = 60 + 18 = 78$

Visual proof

- https://www.youtube.com/watch?v=p9hIwxLmCFk

The Triangular Number of a Product

Identity for integers $n > 1$ and $k > 1$

$$T_{n \cdot k} = T_n \cdot T_k + T_{n-1} \cdot T_{k-1}$$

Proof

$$
\begin{aligned}
T_{nk} &= \frac{nk(nk+1)}{2} = \frac{n^2k^2 + nk}{2} = \frac{2n^2k^2 + 2nk}{4} \\
&= \frac{2n^2k^2 + (n^2k - n^2k) + (nk^2 - nk^2) + 2nk}{4} \\
&= \frac{(n^2k^2 + n^2k + nk^2 + nk) + (n^2k^2 - n^2k - nk^2 + nk)}{4} \\
&= \frac{n^2k^2 + n^2k + nk^2 + nk}{4} + \frac{n^2k^2 - n^2k - nk^2 + nk}{4} \\
&= \frac{(n^2+n)(k^2+k)}{4} + \frac{(n^2-n)(k^2-k)}{4} \\
&= \frac{(n(n+1))(k(k+1))}{4} + \frac{((n-1)n)((k-1)k)}{4} \\
&= \frac{n(n+1)}{2}\frac{k(k+1)}{2} + \frac{(n-1)n}{2}\frac{(k-1)k}{2} \\
&= T_n T_k + T_{n-1} T_{k-1}
\end{aligned}
$$

Odd Squares as Difference of Triangular Numbers

Identity for integer $n \geq 1$

$$(2n+1)^2 = T_{3n+1} - T_n$$

Examples

- $(2 \cdot 3 + 1)^2 = 7^2 = 49 = 55 - 6 = T_{10} - T_3 = T_{3 \cdot 3+1} - T_3$
- $(2 \cdot 4 + 1)^2 = 9^2 = 81 = 91 - 10 = T_{13} - T_4 = T_{3 \cdot 4+1} - T_3$
- $(2 \cdot 5 + 1)^2 = 11^2 = 121 = 136 - 15 = T_{16} - T_5 = T_{3 \cdot 5+1} - T_3$

Visual Proof

- https://www.youtube.com/watch?v=hP5ExUA5P8A

Odd Squares as Difference of Triangular Numbers

Identity for integer $n \geq 1$

$$(2n+1)^2 = T_{3n+1} - T_n$$

Proof

$$\begin{aligned}
T_{3n+1} - T_n &= \frac{(3n+1)(3n+2)}{2} - \frac{n(n+1)}{2} \\
&= \frac{9n^2+9n+2}{2} - \frac{n^2+n}{2} \\
&= \frac{8n^2+8n+2}{2} \\
&= 4n^2+4n+1 \\
&= (2n+1)^2
\end{aligned}$$

Sums of Powers of 9 are Triangular Numbers

Identity for integer $n \geq 1$

$$\sum_{i=0}^{n-1} 9^i = \frac{9^n - 1}{8} = T_{\sum_{i=0}^{n-1} 3^i}$$

Examples

- $1 + 9 = 10 = T_4 = T_{1+3}$
- $1 + 9 + 81 = 91 = T_{13} = T_{1+3+9}$

Visual Proof

- `https://www.youtube.com/watch?v=Ch7GFdsc9pQ`

Sum of the Reciprocals of All Triangular Numbers

Identity and proof

$$
\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{T_n} &= \frac{1}{1} + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \frac{1}{15} + \frac{1}{21} + \cdots \\
&= \sum_{n=1}^{\infty} \frac{1}{\frac{n(n+1)}{2}} \\
&= 2 \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \\
&= 2 \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) \\
&= 2 \left(\left(\frac{1}{1} - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \cdots \right) \\
&= 2
\end{aligned}
$$

Visual proof

- https://www.youtube.com/watch?v=I0juicS3SIc&t=119s

Tetrahedral Numbers

Definition

- The tetrahedral number Te_n counts objects arranged in a four equilateral triangular faces pyramid in which each triangular face has T_n objects.

Illustrations

- https://www.geeksforgeeks.org/tetrahedral-numbers/

Animation

- https://en.wikipedia.org/wiki/File:Pyramid_of_35_spheres_animation.gif

Recursive definition

$$Te_n = \begin{cases} 1 & \text{for } n = 1 \\ Te_{n-1} + T_n & \text{for } n > 1 \end{cases}$$

Tetrahedral Numbers

Closed-form expression

$$
\begin{aligned}
Te_n &= \sum_{i=1}^{n} T_i \\
&= T_n + T_{n-1} + T_{n-2} + \cdots + T_2 + T_1 \\
&= \sum_{i=1}^{n} \binom{n+1}{2} \\
&= \binom{n+2}{3} \\
&= \frac{n(n+1)(n+2)}{6}
\end{aligned}
$$

Visual proof

- https://www.youtube.com/watch?v=N0ETyJ5K6j0

Tetrahedral Numbers

Proof sketch

$$\begin{aligned}
\binom{n+2}{3} &= \binom{n+1}{2} + \binom{n+1}{3} \\
&= \binom{n+1}{2} + \binom{n}{2} + \binom{n}{3} \\
&= \binom{n+1}{2} + \binom{n}{2} + \binom{n-1}{2} + \binom{n-1}{3} \\
&\vdots \\
&= \binom{n+1}{2} + \binom{n}{2} + \binom{n-1}{2} + \cdots + \binom{3}{2} + \binom{3}{3} \\
&= \binom{n+1}{2} + \binom{n}{2} + \binom{n-1}{2} + \cdots + \binom{3}{2} + \binom{2}{2} \\
&= \sum_{i=1}^{n} \binom{n+1}{2}
\end{aligned}$$

Tetrahedral Numbers in Pascal's Triangle

1 3 3 **1**

1 **4** 6 **4** 1

1 5 **10** **10** 5 1

1 6 15 **20** 15 6 1

1 7 21 **35** **35** 21 7 1

1 8 28 **56** 70 **56** 28 8 1

1 9 36 **84** 126 126 **84** 36 9 1

1 10 45 **120** 210 252 210 **120** 45 10 1

1 11 55 **165** 330 462 462 330 **165** 55 11 1

1 12 66 **220** 495 792 924 792 495 **220** 66 12 1

1 13 78 **286** 715 1287 1716 1716 1287 715 **286** 78 13 1

1 14 91 **364** 1001 2002 3003 3432 3003 2002 1001 **364** 91 14 1

The Math of "The 12 Days Of Christmas"

https://www.youtube.com/watch?v=fC8W4s6N9HQ

Tetrahedral Numbers as Sum of Products

Theorem for $n \geq 1$

$$\binom{n+2}{3} = \sum_{h=1}^{n} h(n+1-h)$$

The binomial side

$$\binom{n+2}{3} = \frac{(n+2)\cdot(n+1)\cdot n}{3\cdot 2\cdot 1} = \frac{n^3+3n^2+2n}{6}$$

The sum of products side

$$1\cdot n + 2\cdot(n-1) + 3\cdot(n-2) + \cdots + h(n+1-h) + \cdots + (n-2)\cdot 3 + (n-1)\cdot 2 + n\cdot 1$$

Visual proof

- https://www.youtube.com/watch?v=pucFDbdEyuE&t=3s

Tetrahedral Numbers as Sum of Products

Examples

$$\binom{4}{3} = \frac{4 \cdot 3 \cdot 2}{3 \cdot 2 \cdot 1} = 4$$
$$= 1 \cdot 2 + 2 \cdot 1 = 2 + 2 = 4$$

$$\binom{5}{3} = \frac{5 \cdot 4 \cdot 3}{3 \cdot 2 \cdot 1} = 10$$
$$= 1 \cdot 3 + 2 \cdot 2 + 3 \cdot 1 = 3 + 4 + 3 = 10$$

$$\binom{6}{3} = \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} = 20$$
$$= 1 \cdot 4 + 2 \cdot 3 + 3 \cdot 2 + 4 \cdot 1 = 4 + 6 + 6 + 4 = 20$$

$$\binom{7}{3} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35$$
$$= 1 \cdot 5 + 2 \cdot 4 + 3 \cdot 3 + 4 \cdot 2 + 5 \cdot 1 = 5 + 8 + 9 + 8 + 5 = 35$$

Tetrahedral Numbers as Sum of Products

Combinatorial proof: $\binom{n+2}{3} = \sum_{h=1}^{n} h(n+1-h)$

- There are $\binom{n+2}{3}$ ordered triplets $(i < j < k)$ in the set $\{1, \ldots, n+2\}$.
- Fix the middle index j.
- j is neither 1 nor $n+2$ and therefore $2 \leq j \leq n+1$.
- There are $j-1$ ways to select $i \in \{1, 2, \ldots, j-1\}$ and there are $n+2-j$ ways to select $k \in \{j+1, j+2, \ldots, n+2\}$.
- Therefore, the number of triplets (i, j, k) with j as the middle index is $(j-1)(n+2-j)$.
- The total number of triplets is $\sum_{j=2}^{n+1} (j-1)(n+2-j)$.
- Replacing j with $h+1$ implies that this number is $\sum_{h=1}^{n} h(n+1-h)$.

Tetrahedral Numbers as Sum of Products

The combinatorial proof using pseudocodes

- Pseudocode I: The value of c at the end is $\binom{n+2}{3} = \frac{(n+2)(n+1)n}{6}$.
- Pseudocode II: The value of c is $\sum_{j=2}^{n+1} (j-1)(n+2-j)$.

Pseudocode I

```
function f(n)   (* integer n ≥ 1 *)
  c = 0
  for i = 1 to n do
    for j = i + 1 to n + 1 do
      for k = j + 1 to n + 2 do
        c := c + 1
```

Pseudocode II

```
function f(n)   (* integer n ≥ 1 *)
  c = 0
  for j = 2 to n + 1 do
    for i = 1 to j − 1 do
      for k = j + 1 to n + 2 do
        c := c + 1
```

Tetrahedral Numbers as Sum of Products

Example n = 3: lexicographic order vs. proof order

(123)	(123)
(124)	(124)
(125)	(125)
(134)	(134)
(135)	(135)
(145)	(234)
(234)	(235)
(235)	(145)
(245)	(245)
(345)	(345)

Tetrahedral Numbers as Sum of Products

Example n = 4: lexicographic order vs. proof order

(123)	(123)	(234)	(145)
(124)	(124)	(235)	(146)
(125)	(125)	(236)	(245)
(126)	(126)	(245)	(246)
(134)	(134)	(246)	(346)
(135)	(135)	(256)	(346)
(136)	(136)	(345)	(156)
(145)	(234)	(346)	(256)
(146)	(235)	(356)	(356)
(156)	(236)	(456)	(456)

Sum of Squares of Binomial Coefficients

Theorem

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$$

Example n $= 1$

$$\binom{2}{1} = \frac{2}{1} = 2$$

$$\begin{aligned}\binom{2}{1} &= \sum_{k=0}^{1} \binom{1}{k}^2 \\ &= \binom{1}{0}^2 + \binom{1}{1}^2 \\ &= 1^2 + 1^2 = 1 + 1 = 2\end{aligned}$$

Sum of Squares of Binomial Coefficients

Theorem

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$$

Example n $= 2$

$$\begin{aligned}
\binom{4}{2} &= \frac{4 \cdot 3}{2 \cdot 1} = 6 \\
\binom{4}{2} &= \sum_{k=0}^{2} \binom{2}{k}^2 \\
&= \binom{2}{0}^2 + \binom{2}{1}^2 + \binom{2}{2}^2 \\
&= 1^2 + 2^2 + 1^2 = 1 + 4 + 1 = 6
\end{aligned}$$

Sum of Squares of Binomial Coefficients

Theorem

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$$

Example n $= 3$

$$\binom{6}{3} = \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} = 20$$

$$\begin{aligned}\binom{6}{3} &= \sum_{k=0}^{3} \binom{3}{k}^2 \\ &= \binom{3}{0}^2 + \binom{3}{1}^2 + \binom{3}{2}^2 + \binom{3}{3}^2 \\ &= 1^2 + 3^2 + 3^2 + 1^2 = 1 + 9 + 9 + 1 = 20\end{aligned}$$

Sum of Squares of Binomial Coefficients

Theorem

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$$

Example n $= 4$

$$\binom{8}{4} = \frac{8 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 3 \cdot 2 \cdot 1} = 70$$

$$\begin{aligned}
\binom{8}{4} &= \sum_{k=0}^{4} \binom{4}{k}^2 \\
&= \binom{4}{0}^2 + \binom{4}{1}^2 + \binom{4}{2}^2 + \binom{4}{3}^2 + \binom{4}{4}^2 \\
&= 1^2 + 4^2 + 6^2 + 4^2 + 1^2 = 1 + 16 + 36 + 16 + 1 = 70
\end{aligned}$$

Sum of Squares of Binomial Coefficients

Combinatorial proof: $\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$

- There are $\binom{2n}{n}$ ways to select n numbers from the set $\mathcal{S} = \{1, 2, \ldots, 2n\}$.
- Partition the set $\mathcal{S}$ into two disjoint sets $\mathcal{L} = \{1, 2, \ldots, n\}$ and $\mathcal{R} = \{n+1, n+2, \ldots, 2n\}$.
- Every selection of n numbers from $\mathcal{S}$ is a selection of k numbers from $\mathcal{L}$ and $n-k$ numbers from $\mathcal{R}$ for some $0 \le k \le n$.
- For a given k, there are $f(n,k) = \binom{n}{k}\binom{n}{n-k}$ such selections.
- By the symmetry of the binomial coefficient, $f(n,k) = \binom{n}{k}^2$.
- Sum $f(n,k)$ for all $0 \le k \le n$ to get all the selections

$$\binom{2n}{n} = \sum_{k=0}^{n} f(n,k) = \sum_{k=0}^{n} \binom{n}{k}^2$$

Sum of Squares of Binomial Coefficients

Example $2n = 6$: lexicographic order vs. proof order

(123)	(456)	(234)	(124)
(124)	(145)	(235)	(125)
(125)	(146)	(236)	(126)
(126)	(156)	(245)	(134)
(134)	(245)	(246)	(135)
(135)	(246)	(256)	(136)
(136)	(256)	(345)	(234)
(145)	(345)	(346)	(235)
(146)	(346)	(356)	(236)
(156)	(356)	(456)	(123)

Sum of Products

Identity I

$$\binom{r+s}{h} = \sum_{k=0}^{h} \binom{r}{k}\binom{s}{h-k} \quad \text{for given } h \le r \text{ and } h \le s$$

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2 \quad \text{for } h = r = s = n$$

Identity II

$$\binom{n+1}{k+1} = \sum_{m=j}^{n-k+j} \binom{m}{j}\binom{n-m}{k-j} \quad \text{for given } 0 \le j \le k \le n$$

$$\binom{n+1}{k+1} = \sum_{m=k}^{n} \binom{m}{k} \quad \text{for } j = k$$