Discrete Structures: Graphs

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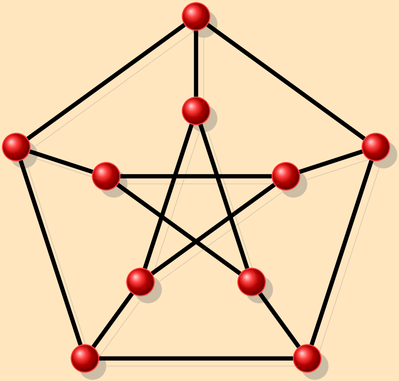
Outline

Graphs

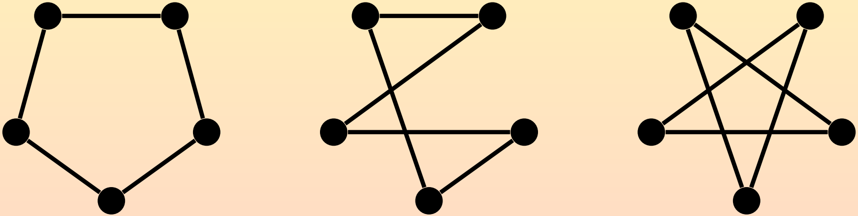
Definition

- A **graph** is a collection of **edges** and **vertices**. Each edge connects two vertices.

The Petersen graph



Different Drawings of the “Same” Graph

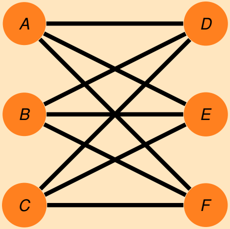


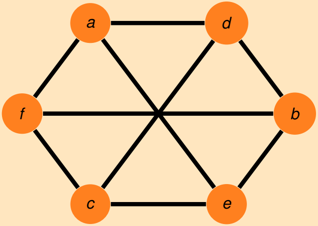
Graph Isomorphism

Definition

- Graph G_1 and graph G_2 are **isomorphic** if there is a **one-to-one function** between their vertices such that, the number of edges joining any two vertices of G_1 is equal to the number of edges joining the corresponding vertices of G_2.

Example





$a \leftrightarrow A \quad b \leftrightarrow B \quad c \leftrightarrow C \quad d \leftrightarrow D \quad e \leftrightarrow E \quad f \leftrightarrow F$

Graph Isomorphism

Both graphs must have the same number of vertices

- Both graphs have 6 edges.
- The graphs are **not isomorphic** because one has 6 vertices while the other has 5 vertices.

Graph Isomorphism

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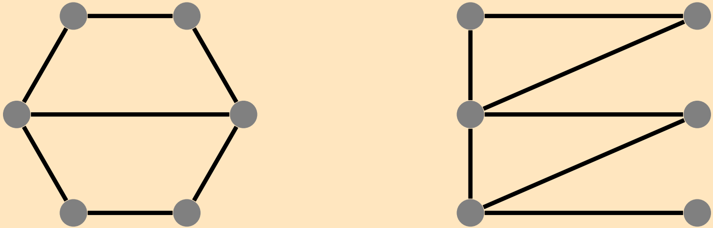
Graph Isomorphism

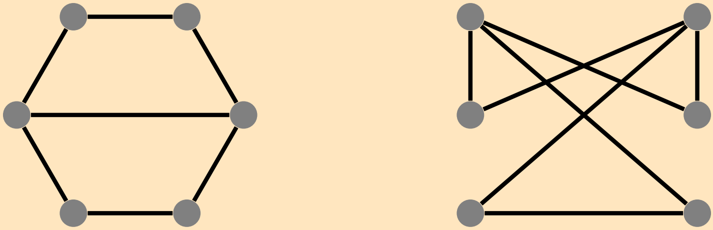
Both graphs must have the same degree sequence



- Both graphs have 6 vertices and 7 edges.
- The graphs are **not isomorphic** because only one of them has a vertex of degree 4.

Graph Isomorphism

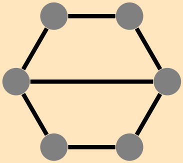
Both graphs must have the same type of connections

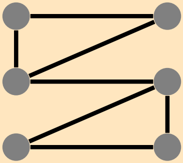


- Both graphs have 6 vertices, 7 edges, and the same degree sequence $(3, 3, 2, 2, 2, 2)$.
- The graphs are **not isomorphic** because the two vertices of degree 3 are connected only in one of them.

Graph Isomorphism

Both graphs must contain the same sub-graphs





- Both graphs have 6 vertices, 7 edges, and the same degree sequence $(3, 3, 2, 2, 2, 2)$.
- In both graphs each vertex with degree 3 is connected to the other vertex of degree 3 and to two vertices of degree 2.
- In both graphs each vertex of degree 2 is connected to another vertex of degree 2 and a vertex of degree 3.
- The graphs are **not isomorphic** because only one of them contains a triangle (a cycle of length 3).

Graph Isomorphism

Problem

- Let G and H be two graphs. Is G **isomorphic** to H?

Algorithm

- Check all possible permutations of the vertices of H and compare them with the vertices of G.
- G and H are isomorphic if at least one of the permutations implies the desired **one-to-one correspondence**.
- This algorithm is very inefficient with an exponential running time.

Hardness

- There is no known efficient algorithm that solves the graph isomorphism problem.
- It is believed that such an algorithm does not exist.

Online Resources

Learn graph theory interactively

- https://d3gt.com/index.html

Online graph editor

- https://csacademy.com/app/graph_editor/

A short introduction

- Definitions and Euler Tour:

  https://youtu.be/2QKjZb9ZKYg?list=PLMyAzUai9V3ox_LDwl54GRkNxovx6NqQX (8:52 min)

- Trees and Traversals:

  https://youtu.be/7OztK4CnsrM?list=PLMyAzUai9V3ox_LDwl54GRkNxovx6NqQX (7:13 min)

Sarada Herke: A Graph Theory Online Course

FAQ

- https://www.youtube.com/playlist?list=PLGxuz-nmYlQOAiikIbmTuj4Lf4QPcOl7G

A comprehensive introductory course with 66 video lectures

- Part I: https://www.youtube.com/playlist?list=PLGxuz-nmYlQOiIOriTXMEoGoybUC3Jmrn
- Part II: https://www.youtube.com/playlist?list=PLGxuz-nmYlQOWynO1-O9SBboVyjSSrmXF
- Part III: https://www.youtube.com/playlist?list=PLGxuz-nmYlQOwe-FPnmy8RA4nzpsygCPx
- Part IV: https://www.youtube.com/playlist?list=PLGxuz-nmYlQOXFjanEQY4WHnPJnAYQSqP
- Part V: https://www.youtube.com/playlist?list=PLGxuz-nmYlQPtH2TgH3MTTkrMYjKtltwk
- Part VI: https://www.youtube.com/playlist?list=PLGxuz-nmYlQMqbct_HCAgSmWEmuHvubXT
- Part VII: https://www.youtube.com/playlist?list=PLGxuz-nmYlQNCcfVYLs9G4dtFJDFUuo5A
- Part VIII: https://www.youtube.com/playlist?list=PLGxuz-nmYlQNtbShUqPRrMA8cQAc45L03
- Part IX: https://www.youtube.com/playlist?list=PLGxuz-nmYlQONimToEreNmISXM808M5Ba
- Part X: https://www.youtube.com/playlist?list=PLGxuz-nmYlQPgIHbqWtgD-F7NnJuqs4fH
- Part XI: https://www.youtube.com/playlist?list=PLGxuz-nmYlQMO2wRhUhV_g6AN3vLN_4X7

Fun with graphs

- https://www.youtube.com/playlist?list=PLGxuz-nmYlQMEo9ULIFc5nRy7pdHZK3vj

MIT Discrete Math lectures: Graph Theory

Part I: Graph Theory and Coloring

- https://ocw.mit.edu/courses/electrical-engineering-and-computer-science/6-042j-mathematics-for-computer-science-fall-2010/video-lectures/lecture-6-graph-theory-and-coloring/

Part II: Matching Problems

- https://ocw.mit.edu/courses/electrical-engineering-and-computer-science/6-042j-mathematics-for-computer-science-fall-2010/video-lectures/lecture-7-matching-problems/

Part III: Minimum Spanning Trees

- https://ocw.mit.edu/courses/electrical-engineering-and-computer-science/6-042j-mathematics-for-computer-science-fall-2010/video-lectures/lecture-8-graph-theory-ii-minimum-spanning-trees/

Part IV: Communication Networks

- https://ocw.mit.edu/courses/electrical-engineering-and-computer-science/6-042j-mathematics-for-computer-science-fall-2010/video-lectures/lecture-9-communication-networks/

Part V: Graph Theory III

- https://ocw.mit.edu/courses/electrical-engineering-and-computer-science/6-042j-mathematics-for-computer-science-fall-2010/video-lectures/lecture-10-graph-theory-iii/

Graph Theory Algorithms by a Google engineer

Description and outline

- `https://www.freecodecamp.org/news/learn-graph-theory-algorithms-from-a-google-engineer/`

Almost 7 hours Video Lecture

- `https://www.youtube.com/watch?v=09_LlHjoEiY&feature=youtu.be` (6:44:39 hours)

Playlist

- `https://www.youtube.com/playlist?list=PLDV1Zeh2NRsDGO4--qE8yH72HFL1Km93P`

Famous Graph Problems

The seven bridges of Königsberg

- https://www.youtube.com/watch?v=nZwSo4vfw6c (4:39 min)

The four color map problem

- https://www.youtube.com/watch?v=ANY7X-_wpNs (2:36 min)
- https://www.youtube.com/watch?v=NgbK43jB4rQ (14:17 min)

The Traveling Salesperson Problem

- https://www.youtube.com/watch?v=l8KBKItQ3T4 (1:15 min)
- https://www.youtube.com/watch?v=SC5CX8drAtU (2:22 min)

Notations

- $G = (V, E)$ – graph.
- $V = \{1, \ldots, n\}$ – set of vertices.
- $E \subseteq V \times V$ – set of edges.
- $e = (u, v) \in E$ – edge.
- $n = |V|$ – number of vertices.
- $m = |E|$ – number of edges.

Directed and Undirected Graphs

Undirected graphs

- The edge (u, v) is the same as the edge (v, u).

Directed graphs (D-graphs)

- The edge $(u \to v)$ is not the same as the edge $(v \to u)$.

The underlying undirected graph of a directed graph

- The edge $(u \to v)$ becomes (u, v).

Directed and Undirected Graphs

Undirected edges

- Vertices u and v are the **endpoints** of the edge (u, v).
- Edge (u, v) is **incident** with vertices u and v.
- Vertices u and v are **neighbors** if edge (u, v) exists. Vertex u is **adjacent** to vertex v and vertex v is **adjacent** to vertex u.
- Vertex u has **degree** d if it has d neighbors.
- Edge (v, v) is a **(self) loop**.
- Edges $e_1 = (u, v)$ and $e_2 = (u, v)$ are **parallel** edges.

Directed and Undirected Graphs

Directed edges

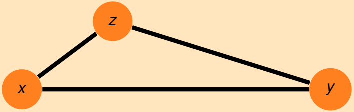
- Vertex u is the **origin** (**initial**) and vertex v is the **destination** (**terminal**) of the directed edge $(u \to v)$.
- Vertex v is the **neighbor** of vertex u if the directed edge $(u \to v)$ exists (but vertex u is not a neighbor of vertex v). Vertex v is **adjacent** to vertex u (but vertex u is not adjacent to vertex v).
- Vertex u has **out-degree** d if it has d neighbors and has **in-degree** d if it is the neighbor of d vertices.
- Edge $(v \to v)$ is a **(self) directed loop**.
- Directed edges $e_1 = (u \to v)$ and $e_2 = (u \to v)$ are **parallel** directed edges (but directed edges $e_1 = (u \to v)$ and $e_2 = (v \to u)$ are not **parallel** directed edges).

Weighted Graphs

Definition

- In **Weighted graphs** there exists a weight function: $w : E \rightarrow \Re$.

The triangle inequality



- For any three edges (x, y), (x, z), and (y, z), the weight function obeys the inequality:

$$w(x, y) \leq w(x, z) + w(y, z)$$

- Example: distances in the plane.

Simple Graphs

Definition

- A **simple** directed or undirected graph is a graph with no parallel edges and no self loops.
- In a simple directed graph both edges: $(u \rightarrow v)$ and $(v \rightarrow u)$ could exist (they are not parallel edges).

Number of edges in simple graphs

- A simple undirected graph has at most $m = \binom{n}{2}$ edges.
- A simple directed graph has at most $m = n(n-1)$ edges.
- A **dense** simple (directed or undirected) graph has “many” edges: $m = \Theta(n^2)$.
- A **sparse** (**shallow**) simple (directed or undirected) graph has “few” edges: $m = \Theta(n)$.

Labeled and Unlabeled Graphs

Definition

- In a **labeled** graph each vertex has a unique label (ID).
  - $*$ Usually the labels are: $1, \ldots, n$.

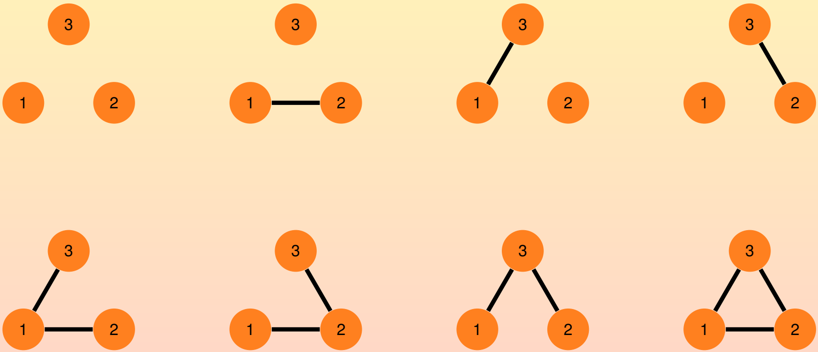
Observation

- There are $2^{\binom{n}{2}}$ **non-isomorphic** labeled graphs with n vertices. Because each possible edge exists or does not exist.

Open problem

- There is no known formula for the number of distinct unlabeled non-isomorphic graphs with $n \geq 1$ vertices.
- There are $1, 2, 4, 11, 34, 156, 1044$ distinct unlabeled non-isomorphic graphs with $n = 1, 2, 3, 4, 5, 6, 7$ vertices.
- There are 24637809253125004524383007491432768 distinct unlabeled non-isomorphic graphs with $n = 20$ vertices.

The 8 Labelled Graphs with $n = 3$ vertices



The 4 Unlabelled Graphs with $n = 3$ Vertices

Paths and Cycles

Paths

- An undirected or directed path $\mathcal{P} = \langle v_0, v_1, \ldots, v_k \rangle$ of length k is an ordered list of vertices such that (v_i, v_{i+1}) or $(v_i \to v_{i+1})$ exists for $0 \leq i \leq k - 1$ and all the edges are different.

Cycles

- An undirected or directed cycle $\mathcal{C} = \langle v_0, v_1, \ldots, v_{k-1}, v_0 \rangle$ of length k is an undirected or directed path that starts and ends with the same vertex.

Simple paths

- In a simple path, directed or undirected, all the vertices are different.

Simple cycles

- In a simple cycle, directed or undirected, all the vertices except $v_0 = v_k$ are different.

Special Paths and Cycles

Euler paths

- An undirected or directed Euler path (**tour**) is a path that traverses all the edges of the graph.

Euler cycles

- An undirected or directed Euler cycle (**circuit**) is a cycle that traverses all the edges of the graph.

Hamiltonian paths

- An undirected or directed Hamiltonian path (**tour**) is a simple path that visits all the vertices of the graph.

Hamiltonian cycles

- An undirected or directed Hamiltonian cycle (**circuit**) is a simple cycle that visits all the vertices of the graph.

Connectivity

Definition

- In a **connected** undirected graph there exists a path between any pair of vertices.

Connected components

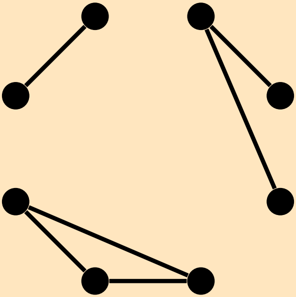
- A connected sub-graph G' is a **connected component** of an undirected graph G if there is no connected sub-graph G'' of G such that G' is also a subgraph of G''.

Corollary

- A connected graph has exactly one connected component.

Connectivity

A graph with three connected components



Strong Connectivity

Definition

- In a **strongly connected** directed graph there exists a directed path from u to v for any pair of vertices u and v.

Strongly connected components

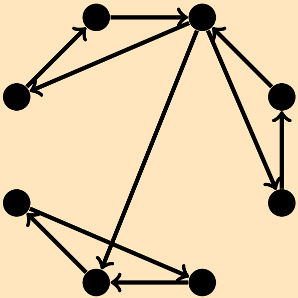
- A strongly connected directed sub-graph G' is a **strongly connected component** of a directed graph G if there is no strongly connected directed sub-graph G'' of G such that G' is also a subgraph of G''.

Corollary

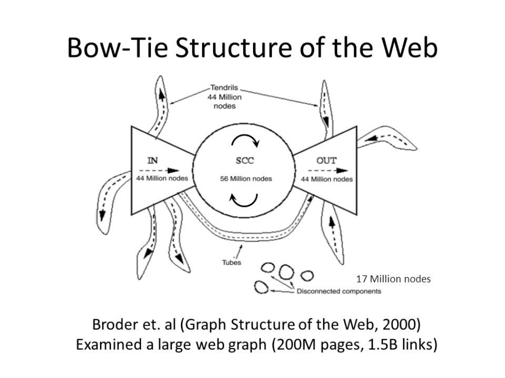
- A strongly connected directed graph has exactly one strongly connected component.

Strongly Connected Directed Graphs

A graph with two strongly connected components



The WEB Graph



Definition

- In the **WEB graph**, every **page** is a vertex and a **hyper-link** from page p to page q is modeled by the directed edge $(p \rightarrow q)$.

Assumptions

- Unless stated otherwise, **usually** a graph is:
  - Simple.
  - Undirected.
  - Unlabelled.
  - Unweighted.
  - Connected.

Forests and Trees

Forests

- Graphs with no cycles.

Trees

- Connected graphs with no cycles.

Trees and Forests

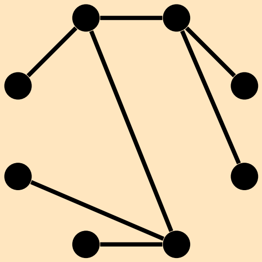
- A tree is a connected forest.
- Each connected component of a forest is a tree.

$n = 1$ and $n = 2$

- For $n = 1$, the singleton vertex is a tree.
- For $n = 2$, the graph with two isolated vertices is a forest and an edge is a tree.

Trees

Example: a tree with 8 vertices



The three characterizations of trees

- A tree is a **connected** graph.
- A tree with n vertices has $\mathbf{n-1}$ edges.
- A tree has **no cycles**.

Trees

Theorem 1: three equivalent definitions

- An undirected and simple graph is a tree if
  - ∗ It is **connected** and has **no cycles**.
  - ∗ It is **connected** and has exactly $\mathbf{m = n - 1}$ edges.
  - ∗ It has **no cycles** and has exactly $\mathbf{m = n - 1}$ edges.

Corollary

- The number of edges in a forest with n vertices and k trees is $m = n - k$.

Theorem 2: three properties

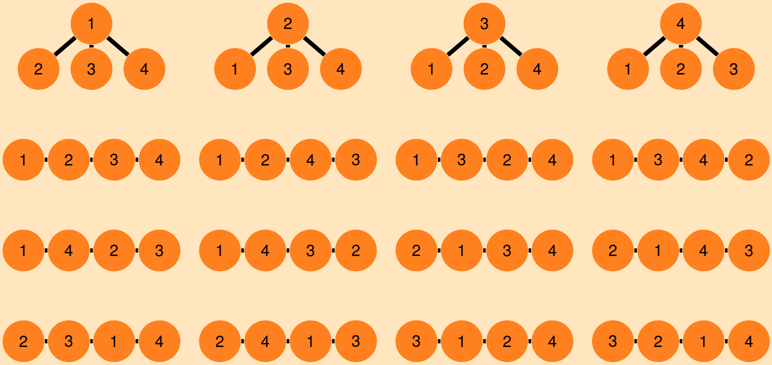
- An undirected and simple graph is a tree if
  - ∗ It is connected and deleting any edge disconnects it.
  - ∗ Any two vertices are connected by exactly one path.
  - ∗ It has no cycles and any new edge forms one cycle.

Counting Labelled Trees

Theorem

- There are n^{n-2} **distinct** labelled n vertices trees.

All labelled with four vertices



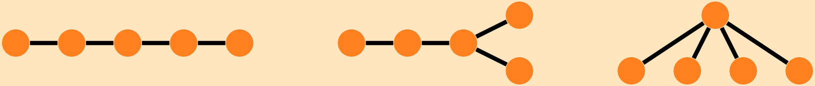
Counting Unlabelled Trees

Open problem

- What is the number of non-isomorphic unlabelled trees with n vertices?

The two unlabelled trees with four vertices

The three unlabelled trees with five vertices

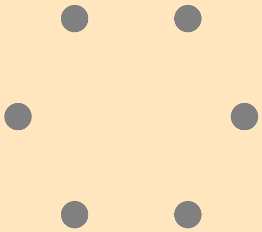


Null Graphs

Definition

- **Null graphs** are graphs with no edges.
- In null graphs $m = 0$.

The null graph with six vertices

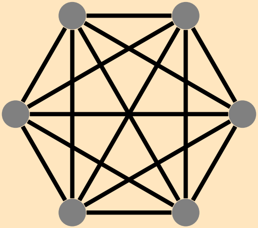


Complete Graphs

Definition

- **Complete graphs** (**cliques**) are graphs with all possible edges.
- In complete graphs $m = \binom{n}{2} = \frac{n(n-1)}{2}$.

The complete graph with six vertices

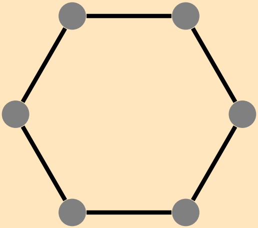


Cycles

Definition

- **Cycles** (**rings**) are connected graphs in which all vertices have degree 2 ($n \geq 3$).
- In cycles $m = n$.

The cycle graph with six vertices

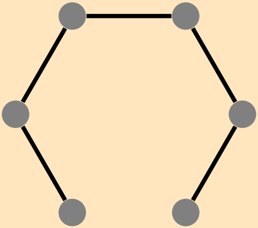


Paths

Definition

- **Paths** are cycles with one edge removed (paths are trees).
- In paths $m = n - 1$.

The path graph with six vertices

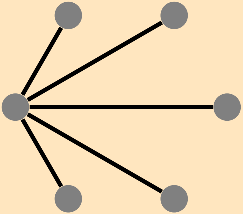


Stars

Definition

- **Stars** are graphs with one root that is connected to $n-1$ leaves (stars are trees).
- The degree of the root is $n-1$ and the degree of each leaf is 1.
- In stars $m = n - 1$.

The star graph with six vertices

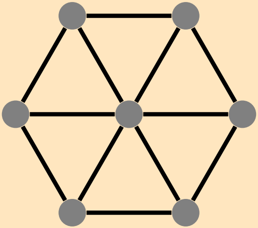


Wheels

Definition

- **Wheels** are stars in which all the $n-1$ leaves form a cycle.
- In wheels $m = 2n - 2$ for $n \geq 4$.

The wheel graph with seven vertices



Bipartite Graphs

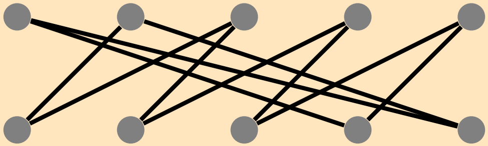
Definition

- The vertices of a **bipartite graph** $G = (V, E)$ are partitioned into two disjoint sets $V = X \cup Y$.
- Each edge in E is incident to one vertex from X and one vertex from Y.

Observation

- A graph is bipartite **iff** each cycle in the graph is of even length.

A bipartite graphs with 10 vertices

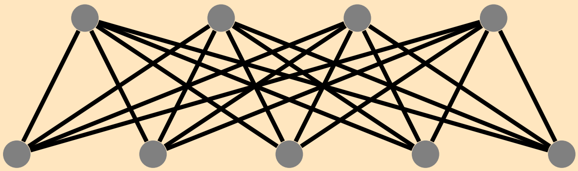


Complete Bipartite Graphs

Definition

- A **complete bipartite graph** is a bipartite graph in which the set X has x vertices, the set Y has y vertices, and all possible $x \cdot y$ edges exist.

A complete bipartite graph with $x = 4$ and $y = 5$ vertices

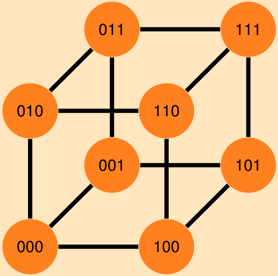


Hyper-Cubes

Definition

- The **Hyper-Cube** graph H_k has $n = 2^k$ vertices representing all the 2^k binary sequences of length k.
- Two vertices in H_k are adjacent if their corresponding sequences differ by exactly one bit.

A hyper-cube graph with 8 vertices



Hyper-Cubes

Observation

- Hyper-Cubes are bipartite graphs.

Proof

- X: The set of all the vertices with even number of 1 in their binary representation.
- Y: The set of all the vertices with odd number of 1 in their binary representation.
- Any edge connects two vertices that differ by one bit and therefore one is from the set X and one is from the set Y.

Planar Graphs

Definition

- **Planar graphs** are graphs that can be drawn on the plane such that edges do not cross each other.

Theorem

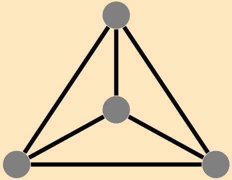
- A graph is planar **iff** it does not have sub-graphs **homeomorphic** to the complete graph with 5 vertices and the complete $\langle 3, 3\rangle$ bipartite graph.

Theorem

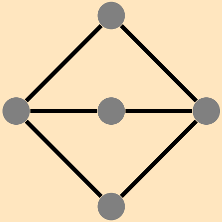
- Every planar graph can be drawn with straight lines.

Small Planar Graphs

The complete graph with 4 vertices

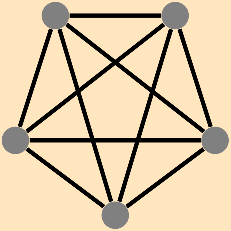


The complete $\langle 2, 3 \rangle$ bipartite graph

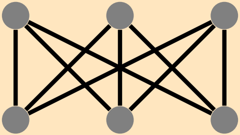


Small Non-Planar Graphs

The complete graph with 5 vertices



The complete $\langle 3, 3 \rangle$ bipartite graph

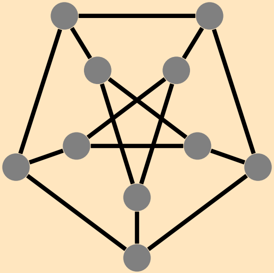


Regular Graphs

Definition

- In **Δ-regular graphs** the degree of each vertex is exactly Δ.
- In Δ-regular graphs $m = \frac{\Delta \cdot n}{2}$.

The 3-regular Petersen graph



Random Graphs

Definition I

- The **random graph** $R(n, p)$ has n vertices and each of the possible $\frac{n(n-1)}{2}$ edges exists with probability $0 \leq p \leq 1$.

Observation

- The expected number of edges in $R(n, p)$ is $p\frac{n(n-1)}{2}$.

Definition II

- The **random graph** $R(n, m)$ is randomly selected with a **uniform distribution** over all graphs with n vertices and m edges.

Remarks

- Both definitions share many properties but they are not equivalent.
- There are many other random graphs models.

Social Graphs

Definition

- A **social graph** contains all the **friendship** relations (edges) among n **people** (vertices).

Propositions

- In any group of $n \geq 2$ people, there are 2 people with the same number of friends in the group.
- There exists a group of 5 people for which no 3 are mutual friends and no 3 are mutual strangers.
- Every group of 6 people contains either three mutual friends or three mutual strangers.

Data structure for Graphs

Goal

- Represent the vertices and edges of the graph **efficiently**.

Representations

- **Adjacency lists:** $\Theta(n+m)$ memory size.
- **Adjacency matrix:** $\Theta(n^2)$ memory size.
- **Incidence matrix:** $\Theta(n \cdot m)$ memory size.

The Adjacency Lists Representation

Definition

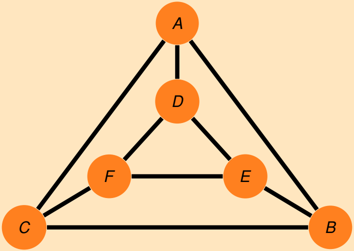
- Each vertex is associated with a linked list consisting of all of its neighbors.
- In a directed graph there are two lists: an incoming list and an outgoing list.
- In a weighted graph each record in the list has an additional field for the weight.

$\Theta(n+m)$-memory

- Undirected graphs: $\sum_v Deg(v) = 2m$
- Directed graphs: $\sum_v OutDeg(v) = \sum_v InDeg(v) = m$

The Adjacency Lists Representation

Example: an undirected graph

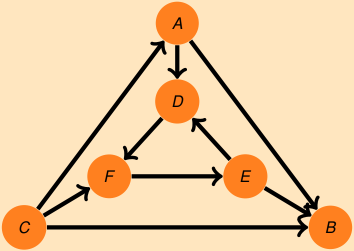


Example: the adjacency lists

$$\begin{array}{ccc} A & \rightarrow & (B, C, D) \\ B & \rightarrow & (A, C, E) \\ C & \rightarrow & (A, B, F) \\ D & \rightarrow & (A, E, F) \\ E & \rightarrow & (B, D, F) \\ F & \rightarrow & (C, D, E) \end{array}$$

The Adjacency Lists Representation

Example: a directed graph



Example: the adjacency lists

$$
\begin{array}{ccccccc}
A & \rightarrow & (B, D) & \qquad & (C) & \rightarrow & A \\
B & \rightarrow & () & & (A, C, E) & \rightarrow & B \\
C & \rightarrow & (A, B, F) & & () & \rightarrow & C \\
D & \rightarrow & (F) & & (A, E) & \rightarrow & D \\
E & \rightarrow & (B, D) & & (F) & \rightarrow & E \\
F & \rightarrow & (E) & & (C, D) & \rightarrow & F
\end{array}
$$

The Adjacency Matrix Representation

Definition

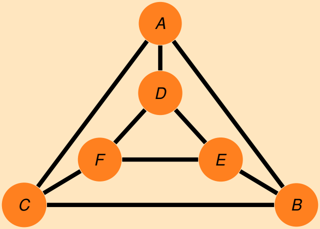
- A matrix A of size $n \times n$:
  - $A[u, v] = 1$ if (u, v) or $(u \rightarrow v)$ is an edge.
  - $A[u, v] = 0$ if (u, v) or $(u \rightarrow v)$ is not an edge.
- In simple graphs: $A[u, u] = 0$
- In undirected graphs: $A[u, v] = A[v, u]$
- In weighted graphs: $A[u, v] = w(u, v)$

$\Theta(n^2)$-memory

- Independent of m that could be $o(n^2)$ and even $O(n)$.

The Adjacency Matrix Representation

Example: an undirected graph

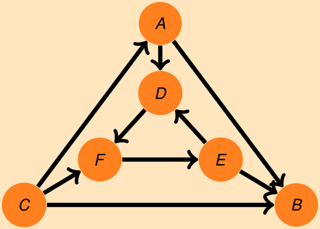


Example: the adjacency matrix

	A	B	C	D	E	F
A	0	1	1	1	0	0
B	1	0	1	0	1	0
C	1	1	0	0	0	1
D	1	0	0	0	1	1
E	0	1	0	1	0	1
F	0	0	1	1	1	0

The Adjacency Matrix Representation

Example: a directed graph



Example: the adjacency matrix

	A	B	C	D	E	F
A	0	1	0	1	0	0
B	0	0	0	0	0	0
C	1	1	0	0	0	1
D	0	0	0	0	0	1
E	0	1	0	1	0	0
F	0	0	0	0	1	0

The Incidence Matrix Representation

Definition

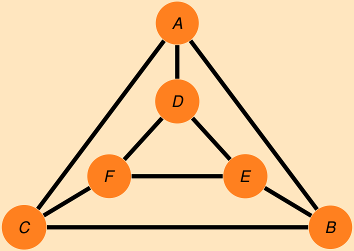
- A matrix A of size $n \times m$:
  - $A[v, e] = 1$ if undirected edge e is incident with v.
  - $A[u, e] = -1$ and $A[v, e] = 1$ for a directed edge $u \to v$.
  - Otherwise $A[v, e] = 0$.
- In simple graphs all the columns are different and each contains exactly two non-zero entries.
- In weighted undirected graphs: $A[v, e] = w(e)$ if edge e is incident with vertex v.

$\Theta(n \cdot m)$-memory

- The memory size depends on the number of edges.

The Incidence Matrix Representation

Example: an undirected graph

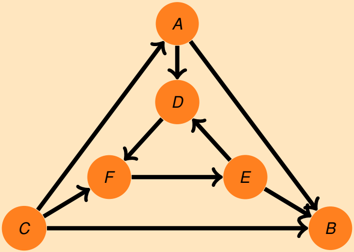


Example: the incidence matrix

	(A,B)	(A,C)	(A,D)	(B,C)	(B,E)	(C,F)	(D,E)	(D,F)	(E,F)
A	1	1	1	0	0	0	0	0	0
B	1	0	0	1	1	0	0	0	0
C	0	1	0	1	0	1	0	0	0
D	0	0	1	0	0	0	1	1	0
E	0	0	0	0	1	0	1	0	1
F	0	0	0	0	0	1	0	1	1

The Incidence Matrix Representation

Example: a directed graph



Example: the incidence matrix

	(A, B)	(A, C)	(A, D)	(B, C)	(B, E)	(C, F)	(D, E)	(D, F)	(E, F)
A	-1	1	-1	0	0	0	0	0	0
B	1	0	0	1	1	0	0	0	0
C	0	-1	0	-1	0	-1	0	0	0
D	0	0	1	0	0	0	1	-1	0
E	0	0	0	0	-1	0	-1	0	1
F	0	0	0	0	0	1	0	1	-1

Which Data Structure to Choose?

Adjacency matrices

- Simpler to implement and maintain.
- Easy to find out if a graph contains a specific edge.
- Easy to add or delete edges.
- Efficient for dense graphs.

Adjacency lists

- Efficient for sparse graphs.
- Used by algorithms whose complexity depends on m.

Incidence matrices

- Useful for **hypergraphs** in which hyperedges may contain more than two vertices.
- Not efficient for graph algorithms.

Graphic Sequences

Degrees

- The **degree** d_v of vertex v in graph G is the number of neighbors of v in G.

The hand-shaking lemma

- **Lemma:** $\sum_{i=1}^{n} d_i = 2m$.
- **Proof outline:** Each edge **"contributes"** exactly 2 to the sum.
- **Corollary:** The number of odd degree vertices is even.

Graphic sequences

- The **degree sequence** of G is $S = (d_1, \ldots, d_n)$.
- A sequence $S = (d_1, \ldots, d_n)$ is **graphic** if there exists a graph with n vertices whose degree sequence is S.

Examples of Non-Graphic Sequences

$(3, 3, 3, 3, 3, 3, 3)$

- Since the sum of the degrees in any graph must be even.
- There is no 7-vertex 3-regular graph.

$(5, 5, 4, 4, 0)$

- Since there are 5 vertices and therefore the maximum degree could be at most 4.
- The maximum degree in a graph with n vertices is $n - 1$.

$(3, 2, 1, 0)$

- Since there is a vertex with degree 3 and only two additional vertices with a positive degree.

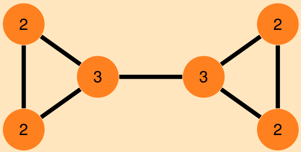
Testing if Sequences are Graphic

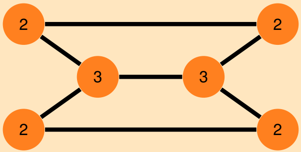
Observation

- Each graph is associated with a degree sequence while a degree sequence might be associated with more than one **non-isomorphic** graph.

Example

- The degree sequence of both graphs below is $(3, 3, 2, 2, 2, 2)$.
- The two graphs are **not isomorphic** because one of them has two cycles of size 3 while the other has two cycles of size 4.





Testing if Sequences are Graphic

Theorem (Erdős-Gallai)

- For $n \geq 1$, a sequence $(d_1 \geq d_2 \geq \cdots \geq d_n)$ of n non-negative integers is **graphic** if the following two conditions hold:
  - $d_1 + d_2 + \cdots + d_n$ is even.
  - for $1 \leq k \leq n$:

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min\{d_i, k\} .$$

Complexity

- A **dynamic programming** based algorithm can check all the n inequalities with complexity $\Theta(n)$.

Remark

- The theorem does not provide a **realization** graph if the sequence is graphic.

Graphic Sequence for Trees

Theorem

- For $n \geq 2$, a sequence $(d_1, d_2, \ldots, d_n)$ of n positive integers is a degree sequence of a tree **iff**

$$\sum_{i=1}^{n} d_i = 2n - 2$$

Proof

$\Rightarrow$ A tree has $n - 1$ edges. By the **hand-shaking lemma** the sum of the degrees in a tree is $2n - 2$.

$\Leftarrow$ By induction on n.

Online resource

- https://www.youtube.com/watch?v=cCG4_mj9TgM

Graphic Sequences – Observations

Observation I

- The sequence $(0, 0, \ldots, 0)$ of length n is graphic since it represents the null graph with n vertices.

Observation II

- $d_1 \leq n - 1$ in a graphic sequence $S = (d_1 \geq \cdots \geq d_n)$.

Observation III

- $d_{d_1+1} > 0$ in a graphic sequence $S = (d_1 \geq \cdots \geq d_n)$ of a non-null graph.
- Equivalently, if $d_1 > 0$ then there are at least $d_1 + 1$ non-zeros in S.

Transformation

Definition

- Let $S = (d_1 \geq d_2 \geq \cdots \geq d_n)$.
- Then $f(S) = (d_2 - 1 \geq \cdots \geq d_{d_1+1} - 1, d_{d_1+2} \geq \cdots \geq d_n)$.

Examples

$$S = (5, 4, 3, 3, 2, 1, 1, 1) \implies f(S) = (3, 2, 2, 1, 0, 1, 1)$$
$$S = (6, 6, 6, 3, 3, 2, 2, 2, 2) \implies f(S) = (5, 5, 2, 2, 1, 1, 2, 2)$$

Remarks

- The transformation can be applied only if both **Observation II** and **Observation III** hold.
- The transformation does not change S if **Observation I** holds.

Graphic Sequences

Theorem (Havel-Hakimi)

- $S = (d_1 \geq \cdots \geq d_n)$ is graphic **iff** $f(S)$ is graphic.

Proof outline

$\Leftarrow$ To get a graphic representation for S, add a vertex of degree d_1 to the graphic representation of $f(S)$ and connect this vertex to all vertices whose degrees in $f(S)$ are smaller by 1 than those in S.

$\Rightarrow$ To get a graphic representation for $f(S)$, omit a vertex of degree d_1 from the graphic representation of S. Make sure (**how?**) that this vertex is connected to the vertices whose degrees are $d_2, \ldots, d_{d_1+1}$.

Online resources

- `https://www.youtube.com/watch?v=aNKO4ttWmcU`
- `https://www.youtube.com/watch?v=iQJ1PFZ4gh0`

Algorithm to Test if a Sequence is Graphic

Algorithm

Graphic($S = (d_1 \geq \cdots \geq d_n \geq 0)$)

case $d_1 = 0$ return(TRUE)

case $d_1 \geq n$ return(FALSE)

case $d_{d_1+1} = 0$ return(FALSE)

otherwise return **Graphic(Sort($f(S)$))**

Termination

- The sequence's length is reduced by 1 after each recursive call. Thus, the algorithm terminates after at most $n - 1$ recursive calls.

Correctness

- **Observation I** implies the first case.
- **Observation II** implies the second case.
- **Observation III** implies the third case.
- The **theorem** justifies the recursion.

Constructing the Realization Graph

Setting

- Let $S = (d_1 \geq d_2 \geq \cdots \geq d_n)$ be a graphic sequence.
- Let the vertices of S be $v_1, v_2, \ldots, v_n$ where the degree of v_i should be d_i.

Construction outline

- Initially there are no edges in the graph.
- In each round
  - ∗ Let d be the degree of one of the highest degree vertices v_i.
  - ∗ Let $v_{i_1}, v_{i_2}, \ldots, v_{i_d}$ be the next d vertices with the highest degrees.
  - ∗ These vertices are the new neighbors of v_i.
  - ∗ For all $1 \leq j \leq d$, add the edge (v_i, v_{i_j}) to the graph.
  - ∗ Update the degree of v_i to be 0 and reduce the degrees of each one of $v_{i_1}, \ldots, u_{i_d}$ by one.
- Terminate when the degree of all vertices is 0.

Example I

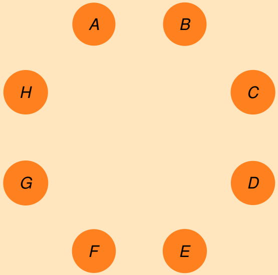
Initial sequence

- $(A, B, C, D, E, F, G, H) = (4, 4, 3, 2, 2, 2, 2, 1)$.

Tiebreaker rules

- Selecting the neighbors: by the alphabetical order from A to H.

Initial graph

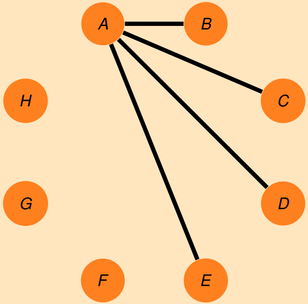


Example I

Round 1

- Sequence before: $(A, B, C, D, E, F, G, H) = (4, 4, 3, 2, 2, 2, 2, 1)$.
- New edges: A is connected to B, C, D, and E.
- Sequence after: $(A, B, C, D, E, F, G, H) = (0, 3, 2, 1, 1, 2, 2, 1)$.

Graph after Round 1

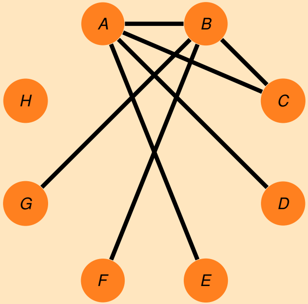


Example I

Round 2

- Sequence before: $(A, B, C, D, E, F, G, H) = (0, 3, 2, 1, 1, 2, 2, 1)$.
- New edges: B is connected to C, F, and G.
- Sequence after: $(A, B, C, D, E, F, G, H) = (0, 0, 1, 1, 1, 1, 1, 1)$.

Graph after Round 2

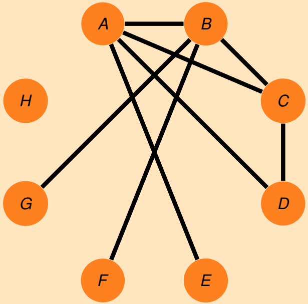


Example I

Round 3

- Sequence before: $(A, B, C, D, E, F, G, H) = (0, 0, 1, 1, 1, 1, 1, 1)$.
- New edge: C is connected to D.
- Sequence after: $(A, B, C, D, E, F, G, H) = (0, 0, 0, 0, 1, 1, 1, 1)$.

Graph after Round 3

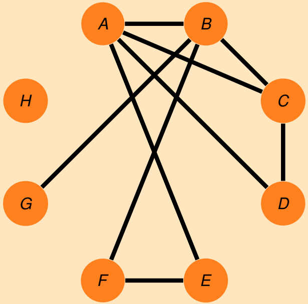


Example I

Round 4

- Sequence before: $(A, B, C, D, E, F, G, H) = (0, 0, 0, 0, 1, 1, 1, 1)$.
- New edge: E is connected to F.
- Sequence after: $(A, B, C, D, E, F, G, H) = (0, 0, 0, 0, 0, 0, 1, 1)$.

Graph after Round 4

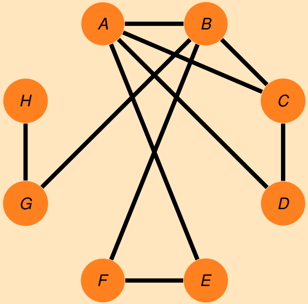


Example I

Round 5

- Sequence before: $(A, B, C, D, E, F, G, H) = (0, 0, 0, 0, 0, 0, 1, 1)$.
- New edge: G is connected to H.
- Sequence after: $(A, B, C, D, E, F, G, H) = (0, 0, 0, 0, 0, 0, 0, 0)$.

Graph after Round 5

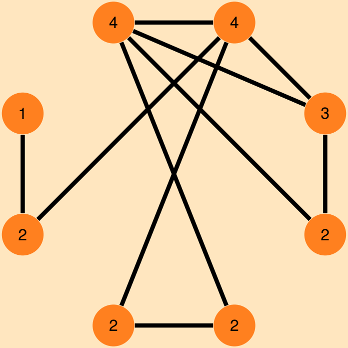


Example I

The graphic sequence

- $(4, 4, 3, 2, 2, 2, 2, 1)$

The realization graph



A Generalization

Algorithm

- Call the vertex that is selected in each round the **pivot** vertex.
- The algorithm works for any vertex being the **pivot** vertex as long as it is connected to the highest degree vertices.

Remarks

- Different selections of **pivot** vertices may lead to different non-isomorphic realizations.
- Different **tiebreaker rules** may lead to different non-isomorphic realizations.
- However, not all the graphs can be realized by this algorithm.

Example II

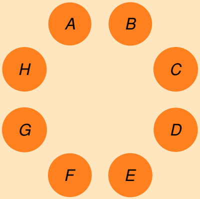
Initial sequence

- $(A, B, C, D, E, F, G, H) = (4, 4, 3, 2, 2, 2, 2, 1)$.

Tiebreaker rules

- Selecting the pivot: one of the smallest degree vertices by the alphabetical order from H to A.
- Selecting the neighbors: by the alphabetical order from A to H.

Initial graph

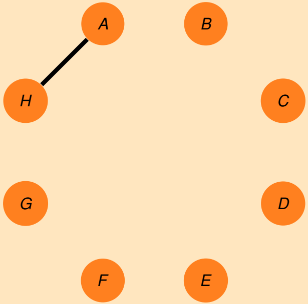


Example II

Round 1

- Sequence before: $(A, B, C, D, E, F, G, H) = (4, 4, 3, 2, 2, 2, 2, 1)$.
- New edge: the pivot H is connected to A.
- Sequence after: $(A, B, C, D, E, F, G, H) = (3, 4, 3, 2, 2, 2, 2, 0)$.

Graph after Round 1

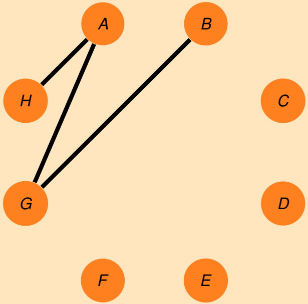


Example II

Round 2

- Sequence before: $(A, B, C, D, E, F, G, H) = (3, 4, 3, 2, 2, 2, 2, 0)$.
- New edges: the pivot G is connected to B and A.
- Sequence after: $(A, B, C, D, E, F, G, H) = (2, 3, 3, 2, 2, 2, 0, 0)$.

Graph after Round 2

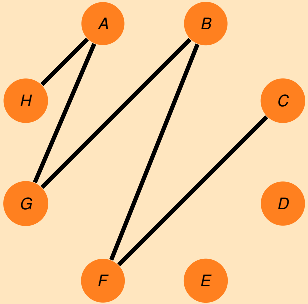


Example II

Round 3

- Sequence before: $(A, B, C, D, E, F, G, H) = (2, 3, 3, 2, 2, 2, 0, 0)$.
- New edges: the pivot F is connected to B and C.
- Sequence after: $(A, B, C, D, E, F, G, H) = (2, 2, 2, 2, 2, 0, 0, 0)$.

Graph after Round 3

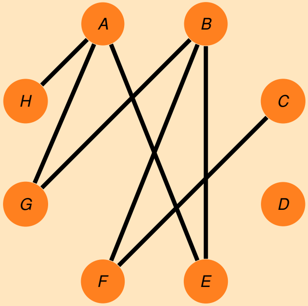


Example II

Round 4

- Sequence before: $(A, B, C, D, E, F, G, H) = (2, 2, 2, 2, 2, 0, 0, 0)$.
- New edges: the pivot E is connected to A and B.
- Sequence after: $(A, B, C, D, E, F, G, H) = (1, 1, 2, 2, 0, 0, 0, 0)$.

Graph after Round 4

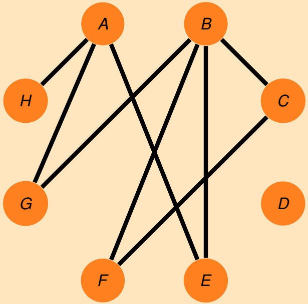


Example II

Round 5

- Sequence before: $(A, B, C, D, E, F, G, H) = (1, 1, 2, 2, 0, 0, 0, 0)$.
- New edge: the pivot B is connected to C.
- Sequence after: $(A, B, C, D, E, F, G, H) = (1, 0, 1, 2, 0, 0, 0, 0)$.

Graph after Round 5

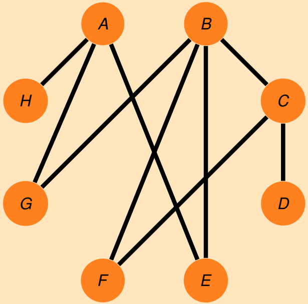


Example II

Round 6

- Sequence before: $(A, B, C, D, E, F, G, H) = (1, 0, 1, 2, 0, 0, 0, 0)$.
- New edge: the pivot C is connected to D.
- Sequence after: $(A, B, C, D, E, F, G, H) = (1, 0, 0, 1, 0, 0, 0, 0)$.

Graph after Round 6

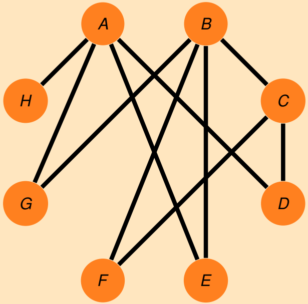


Example II

Round 7

- Sequence before: $(A, B, C, D, E, F, G, H) = (1, 0, 0, 1, 0, 0, 0, 0)$.
- New edge: the pivot D is connected to A.
- Sequence after: $(A, B, C, D, E, F, G, H) = (0, 0, 0, 0, 0, 0, 0, 0)$.

Graph after Round 7

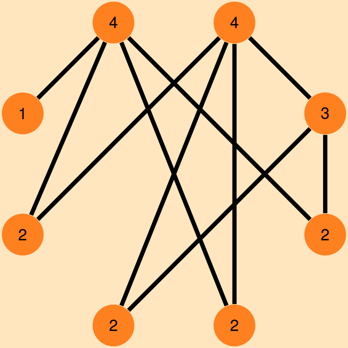


Example II

The graphic sequence

- $(4, 4, 3, 2, 2, 2, 2, 1)$

The realization graph



Example III

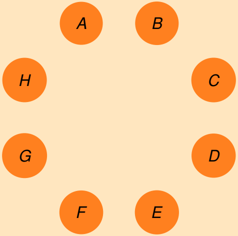
Initial sequence

- $(A, B, C, D, E, F, G, H) = (4, 4, 3, 2, 2, 2, 2, 1)$.

Tiebreaker rules

- Selecting the pivot: arbitrarily.
- Selecting the neighbors: by the alphabetical order from H to A.

Initial graph

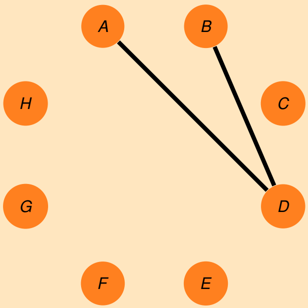


Example III

Round 1

- Sequence before: $(A, B, C, D, E, F, G, H) = (4, 4, 3, 2, 2, 2, 2, 1)$.
- New edges: the pivot D is connected to B and A.
- Sequence after: $(A, B, C, D, E, F, G, H) = (3, 3, 3, 0, 2, 2, 2, 1)$.

Graph after Round 1

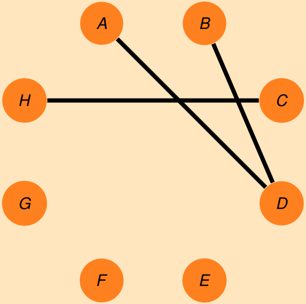


Example III

Round 2

- Sequence before: $(A, B, C, D, E, F, G, H) = (3, 3, 3, 0, 2, 2, 2, 1)$.
- New edge: the pivot H is connected to C.
- Sequence after: $(A, B, C, D, E, F, G, H) = (3, 3, 2, 0, 2, 2, 2, 0)$.

Graph after Round 2

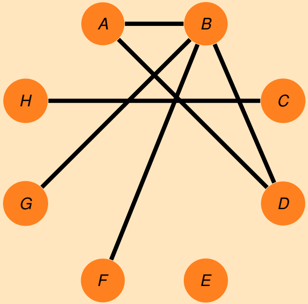


Example III

Round 3

- Sequence before: $(A, B, C, D, E, F, G, H) = (3, 3, 2, 0, 2, 2, 2, 0)$.
- New edges: the pivot B is connected to A, G, and F.
- Sequence after: $(A, B, C, D, E, F, G, H) = (2, 0, 2, 0, 2, 1, 1, 0)$.

Graph after Round 3

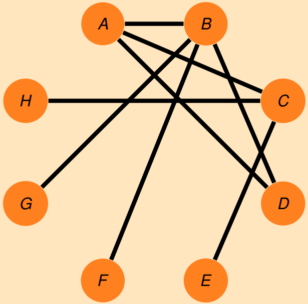


Example III

Round 4

- Sequence before: $(A, B, C, D, E, F, G, H) = (2, 0, 2, 0, 2, 1, 1, 0)$.
- New edges: the pivot C is connected to E and A.
- Sequence after: $(A, B, C, D, E, F, G, H) = (1, 0, 0, 0, 1, 1, 1, 0)$.

Graph after Round 4

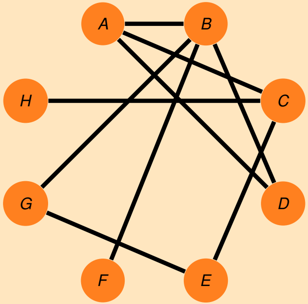


Example III

Round 5

- Sequence before: $(A, B, C, D, E, F, G, H) = (1, 0, 0, 0, 1, 1, 1, 0)$.
- New edge: the pivot E is connected to G.
- Sequence after: $(A, B, C, D, E, F, G, H) = (1, 0, 0, 0, 0, 1, 0, 0)$.

Graph after Round 5

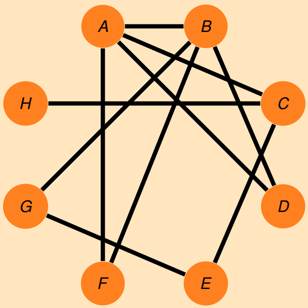


Example III

Round 6

- Sequence before: $(A, B, C, D, E, F, G, H) = (1, 0, 0, 0, 0, 1, 0, 0)$.
- New edge: the pivot A is connected to F.
- Sequence after: $(A, B, C, D, E, F, G, H) = (0, 0, 0, 0, 0, 0, 0, 0)$.

Graph after Round 6

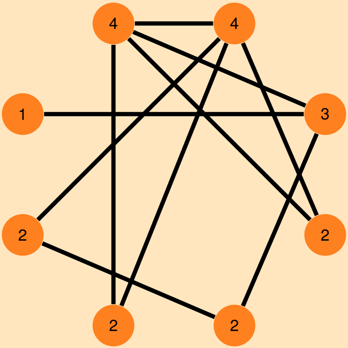


Example III

The graphic sequence

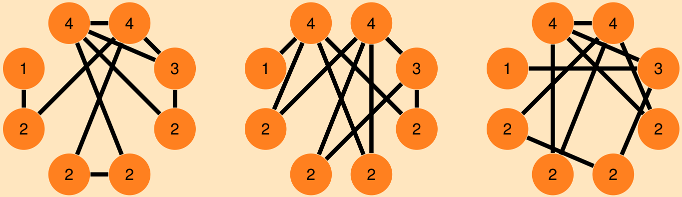
- $(4, 4, 3, 2, 2, 2, 2, 1)$

The realization graph



The Three Realizations Are Not Isomorphic

The realizations



Two differences

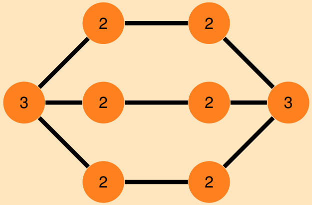
- The degree-1 vertex is connected to a degree-2 vertex in the left realization, to a degree-4 vertex in the middle realization, and to a degree-3 vertex in the right realization.
- The two degree-4 vertices are connected and share only one neighbor in the left realization, the two degree-4 vertices are not connected in the middle realization, and the two degree-4 vertices are connected and share two neighbors in the right realization.

Not All Graphs Can be Realized by the Algorithm

The degree sequence

- $(3, 3, 2, 2, 2, 2, 2, 2)$

An impossible realization



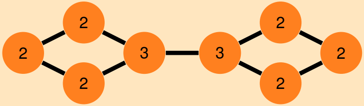
- If the first pivot is a degree-3 vertex, it must be connected to the other degree-3 vertex.
- If the first pivot is a degree-2 vertex, it must have the two degree-3 vertices as its neighbors.

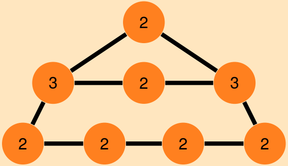
Not All Graphs Can be Realized by the Algorithm

The degree sequence

- $(3, 3, 2, 2, 2, 2, 2, 2)$

Two possible realizations





The two realizations are not isomorphic

- The two degree-3 vertices are neighbors in the left realization while they are not neighbors in the right realization.