# Discrete Structures: Induction

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# The Principle of Induction

## The principle

- Let $P_n$ be a statement about all positive integers $n = 1, 2, 3, \ldots$
- If the following hold:
  - **Induction base:** $P_1$ is true
  - **Inductive step:** For all integers $k \geq 1$, if $P_k$ is true then $P_{k+1}$ is true
- Then $P_n$ is true for all integers $n \geq 1$
- The assumption "$P_k$ is true" is the **induction hypothesis**

## Cartoons

- https://lowres.cartooncollections.com/dominos-chain_reactions-black_humor-black_humour-gallows_humor-social-issues-CC39203_low.jpg
- http://crystalclearmaths.com/wp-content/uploads/Domino-Effect.png

# Some Online Resources

- An introductory video in less than 4 minutes:
  https://www.youtube.com/watch?v=bePpPFos0kE
- Introduction in 15 minutes:
  https://www.youtube.com/watch?v=ruBnYcLzVlU
- Sum of the first $n$ integers in 7 minutes:
  https://www.youtube.com/watch?v=dMn5w4_ztSw&feature=youtu.be
- Sum of the first $n$ odd integers in 10 minutes:
  https://www.youtube.com/watch?v=twA6vZgX_U4
- Sum of first $n$ integers of the form $5k - 1$ in 6 minutes:
  https://www.youtube.com/watch?v=IFqna5F0kW8
- $6^n + 4$ is divisible by 5 in 6 minutes:
  https://youtu.be/MpjkLf7lfRA
- Introduction in 8 minutes (from 11:25 to 20:04):
  https://youtu.be/0UgID8C9RvE?list=PLZzHxk_TPOStgPtqRZ6KzmkUQBQ8TSWVX

# Sum of First $n$ Positive Integers

## An identity

$$\sum_{i=1}^{n} i = 1 + 2 + \cdots + (n-1) + n = \frac{n(n+1)}{2}$$

## An equivalent identity

$$\sum_{i=1}^{n-1} i = 1 + 2 + \cdots + (n-2) + (n-1) = \frac{(n-1)n}{2}$$

# Correctness for Small $n$

$$\begin{aligned}
1 &= 1 = \frac{1 \cdot 2}{2} \\
1 + 2 &= 3 = \frac{2 \cdot 3}{2} \\
1 + 2 + 3 &= 6 = \frac{3 \cdot 4}{2} \\
1 + 2 + 3 + 4 &= 10 = \frac{4 \cdot 5}{2} \\
1 + 2 + 3 + 4 + 5 &= 15 = \frac{5 \cdot 6}{2} \\
1 + 2 + 3 + 4 + 5 + 6 &= 21 = \frac{6 \cdot 7}{2} \\
1 + 2 + 3 + 4 + 5 + 6 + 7 &= 28 = \frac{7 \cdot 8}{2} \\
1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 &= 36 = \frac{8 \cdot 9}{2}
\end{aligned}$$

# Proof By Induction

### Notations

- $L(n) = \sum_{i=1}^{n} i = 1 + 2 + \cdots + (n-1) + n$
- $R(n) = \frac{n(n+1)}{2}$

### The induction base: $n = 1$

- $L(1) = R(1)$, because $L(1) = 1$ and $R(1) = \frac{1 \cdot 2}{2} = 1$

### The induction hypothesis: $L(k) = R(k)$ for $k \geq 1$

$$\sum_{i=1}^{k} i = 1 + 2 + \cdots + (k-1) + k = \frac{k(k+1)}{2}$$

# Proof By Induction

**The inductive step:** $L(k+1) = R(k+1)$ **for** $k \geq 1$

$$
\begin{aligned}
L(k+1) &= 1 + 2 + \cdots + k + (k+1) \\
&= L(k) + (k+1) \\
&= R(k) + (k+1) \\
&= \frac{k(k+1)}{2} + (k+1) \\
&= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \\
&= \frac{k(k+1) + 2(k+1)}{2} \\
&= \frac{(k+2)(k+1)}{2} \\
&= \frac{(k+1)((k+1)+1)}{2} \\
&= R(k+1)
\end{aligned}
$$

# Another Proof

## Idea

- Prove that $2L(n) = 2R(n)$ implying $L(n) = R(n)$

## Example

$$
\begin{aligned}
L(4) &= 1+2+3+4 \\
L(4) &= 4+3+2+1 \\
2L(4) &= 5+5+5+5 \\
2L(4) &= 4 \cdot 5 = 20 \\
L(4) &= 20/2 = 10 \\
R(4) &= \frac{4 \cdot 5}{2} = 10
\end{aligned}
$$

# Another Proof

## The General Case

$$\begin{array}{rcl} L(n) & = & 1 + 2 + 3 + \cdots + (n-2) + (n-1) + n \\ L(n) & = & n + (n-1) + (n-2) + \cdots + 3 + 2 + 1 \\ 2L(n) & = & (n+1) + (n+1) + (n+1) + \cdots + (n+1) + (n+1) + (n+1) \\ 2L(n) & = & n(n+1) \\ L(n) & = & \frac{n(n+1)}{2} = R(n) \end{array}$$

## A proof without words

- `https://i.stack.imgur.com/yerzW.png`

# Sum of First $n$ Even Positive Integers

## Identity

$$\sum_{i=1}^{n}(2i) = 2 + 4 + \cdots + 2(n-1) + 2n = n(n+1)$$

## Correctness for small $n$

$$\begin{aligned}
2 &= 2 = 1 \cdot 2 \\
2+4 &= 6 = 2 \cdot 3 \\
2+4+6 &= 12 = 3 \cdot 4 \\
2+4+6+8 &= 20 = 4 \cdot 5 \\
2+4+6+8+10 &= 30 = 5 \cdot 6 \\
2+4+6+8+10+12 &= 42 = 6 \cdot 7 \\
2+4+6+8+10+12+14 &= 56 = 7 \cdot 8 \\
2+4+6+8+10+12+14+16 &= 72 = 8 \cdot 9 \\
2+4+6+8+19+12+14+16+18 &= 90 = 9 \cdot 10 \\
2+4+6+8+10+12+14+16+18+20 &= 110 = 10 \cdot 11
\end{aligned}$$

# Sum of First $n$ Even Positive Integers

## Identity

$$\sum_{i=1}^{n}(2i) = 2+4+\cdots+2(n-1)+2n = n(n+1)$$

## Proof by reduction

$$\begin{aligned}
\sum_{i=1}^{n}(2i) &= 2+4+\cdots+2(n-1)+2n \\
&= 2(1+2+\cdots+(n-1)+n) \\
&= 2\cdot\left(\frac{n(n+1)}{2}\right) \\
&= n(n+1)
\end{aligned}$$

# Proof By Induction

**Notations**

- $L(n) = \sum_{i=1}^{n} (2i) = 2 + 4 + \cdots + 2(n-1) + 2n$
- $R(n) = n(n+1)$

**The induction base:** $n = 1$

- $L(1) = R(1)$, because $L(1) = 2$ and $R(1) = 1 \cdot 2 = 2$

**The induction hypothesis:** $L(k) = R(k)$ **for** $k \geq 1$

$$\sum_{i=1}^{k} (2i) = 2 + 4 + \cdots + 2(k-1) + 2k = k(k+1)$$

# Proof By Induction

**The inductive step:** $L(k+1) = R(k+1)$ **for** $k \geq 1$

$$
\begin{aligned}
L(k+1) &= 2 + 4 + \cdots + 2k + 2(k+1) \\
&= L(k) + 2(k+1) \\
&= R(k) + 2(k+1) \\
&= k(k+1) + 2(k+1) \\
&= (k+2)(k+1) \\
&= (k+1)(k+2) \\
&= R(k+1)
\end{aligned}
$$

# Sum of First $n$ Odd Positive Integers

## Identity

$$\sum_{i=1}^{n}(2i-1) = 1+3+5+\cdots+(2n-3)+(2n-1) = n^2$$

## Correctness for small $n$

$$\begin{aligned}
1 &= 1 = 1^2\\
1+3 &= 4 = 2^2\\
1+3+5 &= 9 = 3^2\\
1+3+5+7 &= 16 = 4^2\\
1+3+5+7+9 &= 25 = 5^2\\
1+3+5+7+9+11 &= 36 = 6^2\\
1+3+5+7+9+11+13 &= 49 = 7^2\\
1+3+5+7+9+11+13+15 &= 64 = 8^2\\
1+3+5+7+9+11+13+15+17 &= 81 = 9^2\\
1+3+5+7+9+11+13+15+17+19 &= 100 = 10^2
\end{aligned}$$

# Sum of First $n$ Odd Positive Integers

## Identity

$$\sum_{i=1}^{n}(2i-1) = 1 + 3 + \cdots + (2n-3) + (2n-1) = n^2$$

## Proof by reduction

$$\begin{aligned}\sum_{i=1}^{n}(2i-1) &= \sum_{i=1}^{n}(2i) - \sum_{i=1}^{n}1 \\ &= n(n+1) - n \\ &= n^2 + n - n \\ &= n^2\end{aligned}$$

## Visual proofs

- https://www.youtube.com/watch?v=IJ0EQCkJCTc
- https://www.youtube.com/watch?v=ZeEOgbLo0Rg
- https://www.youtube.com/watch?v=x3qfFBNRRDg&list=PLZh9gzIvXQUubr38YfIlul9j7_54hXZy_
- https://www.youtube.com/watch?v=jq5AYCgkciE

# Proof By Induction

**Notations**

- $L(n) = \sum_{i=1}^{n}(2i-1) = 1 + 3 + \cdots + (2n-3) + (2n-1)$
- $R(n) = n^2$

**The induction base:** $n = 1$

- $L(1) = R(1)$, because $L(1) = 1$ and $R(1) = 1^2 = 1$

**The induction hypothesis:** $L(k) = R(k)$ **for** $k \geq 1$

$$\sum_{i=1}^{k}(2i-1) = 1 + 3 + \cdots + (2k-3) + (2k-1) = k^2$$

# Proof By Induction

**The inductive step: $L(k+1) = R(k+1)$ for $k \geq 1$**

$$
\begin{aligned}
L(k+1) &= 1 + 3 + \cdots + (2k-1) + (2k+1) \\
&= L(k) + (2k+1) \\
&= R(k) + (2k+1) \\
&= k^2 + (2k+1) \\
&= (k+1)^2 \\
&= R(k+1)
\end{aligned}
$$

# Sum of the First $2n$ Odd Positive Integers

## Identity

- The sum of the first $n$ odd integers is $1/3$ the sum of the next $n$ odd integers:

$$\frac{\sum_{i=1}^{n}(2i-1)}{\sum_{i=n+1}^{2n}(2i-1)} = \frac{1+3+\cdots+(2n-1)}{(2n+1)+(2n+3)+\cdots+(4n-1)} = \frac{1}{3}$$

## Proof by reduction

$$\begin{aligned}
\sum_{i=n+1}^{2n}(2i-1) &= \sum_{i=1}^{2n}(2i-1) - \sum_{i=1}^{n}(2i-1) \\
&= (2n)^2 - n^2 = 4n^2 - n^2 = 3n^2 \\
&= 3\sum_{i=1}^{n}(2i-1)
\end{aligned}$$

## Visual Proofs

- https://youtu.be/MmOTqrtbtFQ?list=PLZh9gzIvXQUubr38YfIlul9j7_54hXZy_
- https://www.youtube.com/watch?v=fTBvVeURb3Q

# Arithmetic Progressions

**Definition**

- A sequence $a_1, a_2, \ldots, a_n$ is an **arithmetic progression** if $a_i - a_{i-1} = d$ for all $2 \le i \le n$ for some real number $d$

**Example:** $a_1 = 5$**,** $d = 3$**, and** $n = 11$

- $5, 8, 11, 14, 17, 20, 23, 26, 29, 32, 35$

**Key observations**

- **Observation 1:** $a_i = a_1 + (i-1)d$ for $1 \le i \le n$
- **Observation 2:** $a_i = a_n - (n-i)d$ for $1 \le i \le n$

**Example:** $a_1 = 5$**,** $d = 3$**, and** $n = 11$

- **Observation 1:** $a_4 = a_1 + (4-1)d = 5 + 3 \cdot 3 = 5 + 9 = 14$
- **Observation 2:** $a_7 = a_{11} - (11-7)d = 35 - 4 \cdot 3 = 35 - 12 = 23$

# Arithmetic Progressions

**Theorem**

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_{n-1} + a_n = \frac{n(a_1+a_n)}{2}$$

$$\frac{\sum_{i=1}^{n} a_i}{n} = \frac{a_1 + a_2 + \cdots + a_{n-1} + a_n}{n} = \frac{a_1+a_n}{2}$$

**The theorem in words version I**

- The sum of all the $n$ numbers in an arithmetic progression of length $n$ is the average between the first and the last numbers multiplied by $n$.

**The theorem in words version II**

- The average of all the $n$ numbers in an arithmetic progression of length $n$ is the average between the first and the last numbers.

# Arithmetic Progressions: $a_1 = 5$, $d = 3$, and $n = 11$

## Sequence

$$5, 8, 11, 14, 17, 20, 23, 26, 29, 32, 35$$

## Sum of all numbers

$$5 + 8 + 11 + 14 + 17 + 20 + 23 + 26 + 29 + 32 + 35 = 220$$

## Average of all numbers

$$220/11 = 20$$

## Average of the first and the last numbers

$$(5 + 35)/2 = 40/2 = 20$$

# Arithmetic Progressions

## Theorem

$$\sum_{i=1}^{n} a_i = \frac{n(a_1+a_n)}{2}$$

## Notation

- Define $S_n = a_1 + a_2 + \cdots + a_{n-1} + a_n$

## Direct proof

$$\begin{aligned}
S_n &= a_1 + (a_1+d) + (a_1+2d) + \cdots + (a_1+(n-2)d) + (a_1+(n-1)d) \\
S_n &= a_n + (a_n-d) + (a_n-2d) + \cdots + (a_n-(n-2)d) + (a_n-(n-1)d) \\
2S_n &= n(a_1+a_n) \\
S_n &= \frac{n(a_1+a_n)}{2}
\end{aligned}$$

# Proof By Induction

## Notations

- $L(n) = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_{n-1} + a_n$
- $R(n) = \frac{n(a_1+a_n)}{2}$

## The induction base: $n = 1$

- $L(1) = R(1)$, because $L(1) = a_1$ and $R(1) = \frac{1 \cdot (a_1+a_1)}{2} = a_1$

## The induction hypothesis: $L(k) = R(k)$ for $k \geq 1$

$$\sum_{i=1}^{k} a_i = \frac{k(a_1+a_k)}{2}$$

# Proof By Induction

**The inductive step:** $L(k+1) = R(k+1)$ **for** $k \geq 1$

$$
\begin{aligned}
L(k+1) &= a_1 + a_2 + \cdots + a_k + a_{k+1} \\
&= L(k) + a_{k+1} \\
&= R(k) + a_{k+1} \\
&= \frac{k(a_1 + a_k)}{2} + a_{k+1} \\
&= \frac{ka_1}{2} + \frac{ka_k}{2} + \frac{2a_{k+1}}{2} \\
&= \frac{ka_1 + a_k}{2} + \frac{2a_{k+1} + (k-1)a_k}{2} \\
&= \frac{ka_1 + (a_1 + (k-1)d)}{2} + \frac{2a_{k+1} + (k-1)(a_{k+1} - d)}{2} \\
&= \frac{(k+1)a_1 + (k-1)d}{2} + \frac{(k+1)a_{k+1} - (k-1)d}{2} \\
&= \frac{(k+1)(a_1 + a_{k+1})}{2} \\
&= R(k+1)
\end{aligned}
$$

# Aruthmetic Progressions Special Cases

**Sum of the first**

# $2^n$ vs. $n^2$

## Theorem

- $2^n > n^2$ for any integer $n \geq 5$

## Why $n \geq 5$?

$$
\begin{aligned}
2^1 = 2 &> 1 = 1^2 \\
2^2 = 4 &= 4 = 2^2 \\
2^3 = 8 &< 9 = 3^2 \\
2^4 = 16 &= 16 = 4^2 \\
2^5 = 32 &> 25 = 5^2 \\
2^6 = 64 &> 36 = 6^2
\end{aligned}
$$

## Other Induction Bases

- For any $m \geq 0$ the **induction base** could be $P_m$ instead of $P_1$
- In this case, the induction is applied to $n = m, m + 1, \ldots$

# $2^n > n^2$: Proof By Induction

**The induction base for $n = 5$**

- $2^5 = 32 > 25 = 5^2$

**The induction hypothesis for $k \geq 5$**

- Assume that $2^k > k^2$

**The inductive step for $k \geq 5$: prove that $2^{k+1} > (k+1)^2$**

$$
\begin{aligned}
2^{k+1} &= 2 \cdot 2^k \\
&> 2k^2 && (* \text{ the induction hypothesis } *) \\
&= k^2 + k^2 \\
&\geq k^2 + 5k && (* \text{ because } k \geq 5 *) \\
&> k^2 + 2k + 1 && (* \text{ because } 3k > 1 *) \\
&= (k+1)^2
\end{aligned}
$$

# A Divisibility Theorem: Proof By Induction

**Theorem**

- $n(n+1)(n+2)$ is divisible by 6 for $n \geq 1$

**The induction base: for $n = 1, 2, 3, 4, 5$**

$$\begin{aligned}
1 \cdot 2 \cdot 3 = 6 &= 1 \cdot 6 \\
2 \cdot 3 \cdot 4 = 24 &= 4 \cdot 6 \\
3 \cdot 4 \cdot 5 = 60 &= 10 \cdot 6 \\
4 \cdot 5 \cdot 6 = 120 &= 20 \cdot 6 \\
5 \cdot 6 \cdot 7 = 210 &= 35 \cdot 6
\end{aligned}$$

**The induction hypothesis for $k \geq 1$**

- Assume that $k(k+1)(k+2)$ is divisible by 6
- That is, $k(k+1)(k+2) = 6q$ for an integer $q$

# A Divisibility Theorem: Proof By Induction

**The inductive step for $k \geq 1$**

$$
\begin{aligned}
(k+1)(k+2)(k+3) &= k(k+1)(k+2) + 3(k+1)(k+2) && (* \text{ algebra } *) \\
&= 6q + 3(k+1)(k+2) && (* \text{ induction hypothesis } *) \\
&= 6q + 6\frac{(k+1)(k+2)}{2} && (* \text{ algebra } *) \\
&= 6q + 6p && (* \text{ either k + 1 or k + 2 is even } *) \\
&= 6(q+p) && (* \text{ Q.E.D. } *)
\end{aligned}
$$

# A Divisibility Theorem: Second Proof

## Theorem

- $n(n+1)(n+2)$ is divisible by 6 for $n \geq 1$

## Proof

- $n$, $n+1$, and $n+2$ are three consecutive integers
- One of them must be divisible by 3
- One (could be the same integer) must be even and therefore is divisible by 2
- Therefore, the product of the three integers must be divisible by $6 = 3 \cdot 2$

# Another Divisibility Theorem

**Theorem**

- $n(n+1)(n+2)$ is divisible by 24 for an even $n \geq 2$

**Small Values of $n$**

$$\begin{aligned} 2 \cdot 3 \cdot 4 &= 24 &= 1 \cdot 24 \\ 4 \cdot 5 \cdot 6 &= 120 &= 5 \cdot 24 \\ 6 \cdot 7 \cdot 8 &= 336 &= 14 \cdot 24 \end{aligned}$$

**Proof**

- $n$, $n+1$, and $n+2$ are three consecutive integers
- One of them must be divisible by 3
- $n$ and $n+2$ are two consecutive even integers
- One of them must be divisible by 4 while the other is divisible by 2
- Therefore, the product of the three integers must be divisible by $24 = 3 \cdot 4 \cdot 2$

# A Set of Size $n \geq 0$ Has $2^n$ Subsets

**The** 1 **subset of** $\mathcal{S} = \emptyset$

$$\emptyset$$

**The** 2 **subsets of** $\mathcal{S} = \{C\}$

$$\emptyset, \{C\}$$

**The** 4 **subsets of** $\mathcal{S} = \{C, R\}$

$$\emptyset, \{C\}, \{R\}, \{C, R\}$$

**The** 8 **subsets of** $\mathcal{S} = \{C, R, B\}$

$$\emptyset, \{C\}, \{R\}, \{B\}, \{C, R\}, \{C, B\}, \{R, B\}, \{C, R, B\}$$

**The** 16 **subsets of** $\mathcal{S} = \{C, R, B, G\}$

$$\emptyset, \{C\}, \{R\}, \{B\}, \{C, R\}, \{C, B\}, \{R, B\}, \{C, R, B\}$$

$$G, \{C, G\}, \{R, G\}, \{B, G\}, \{C, R, G\}, \{C, B, G\}, \{R, B, G\}, \{C, R, B, G\}$$

# A Set of Size $n \geq 0$ Has $2^n$ Subsets

**Proof**

- By induction on the size of the set

**The induction base for $n = 0$ and $n = 1$**

- The only subset of the empty set is the empty set and $2^0 = 1$
- The empty set and the entire set are the only subsets of a set of size 1 and $2^n = 2^1 = 2$

**The induction hypothesis for $k \geq 1$**

- Any set of size $k$ has $2^k$ subsets

**Notations**

- Let $S = \{s_1, s_2, \ldots, s_k, s_{k+1}\}$ be a set of size $k + 1$
- Let $S' = \{s_1, s_2, \ldots, s_k\}$ be the subset of $S$ containing all of its members except $s_{k+1}$

# A Set of Size $n \geq 0$ Has $2^n$ Subsets

## The inductive step for $k \geq 1$: prove that $S$ has $2^{k+1}$ subsets

- By the induction hypothesis, $S'$ has $2^k$ subsets all of them are also subsets of $S$
- Let $R$ be a subset of $S$ that is not a subset of $S'$
  - ∗ It follows that $s_{k+1} \in R$ and that $R' = R \setminus \{s_{k+1}\}$ is a subset of $S'$
- Let $R'$ be a subset of $S'$
  - ∗ Then, $R = R' \cup \{s_{k+1}\}$ is a subset of $S$ that is not a subset of $S'$
- The above two arguments establishes a **one-to-one mapping** from the set of all the subsets that contain $s_{k+1}$ to the set of all the subsets that do not contain $s_{k+1}$
- Therefore, there are also $2^k$ subsets of $S$ that contain $s_{k+1}$
- Since a subset of $S$ either contains $s_{k+1}$ or does not contain $s_{k+1}$, it follows that the number of subsets of $S$ is $2^k + 2^k = 2^{k+1}$

# Example: $\mathcal{S} = \{C, R, B, G, M\}$

**Matching the 16 subsets without $M$ to the 16 subsets with $M$**

| | | | | | |
|---|---|---|---|---|---|
| $\emptyset$ | $\longleftrightarrow$ | $\{M\}$ | $\{R, B\}$ | $\longleftrightarrow$ | $\{R, B, M\}$ |
| $\{C\}$ | $\longleftrightarrow$ | $\{C, M\}$ | $\{R, G\}$ | $\longleftrightarrow$ | $\{R, G, M\}$ |
| $\{R\}$ | $\longleftrightarrow$ | $\{R, M\}$ | $\{B, G\}$ | $\longleftrightarrow$ | $\{B, G, M\}$ |
| $\{B\}$ | $\longleftrightarrow$ | $\{B, M\}$ | $\{C, R, B\}$ | $\longleftrightarrow$ | $\{C, R, B, M\}$ |
| $\{G\}$ | $\longleftrightarrow$ | $\{G, M\}$ | $\{C, R, G\}$ | $\longleftrightarrow$ | $\{C, R, G, M\}$ |
| $\{C, R\}$ | $\longleftrightarrow$ | $\{C, R, M\}$ | $\{C, B, G\}$ | $\longleftrightarrow$ | $\{C, B, G, M\}$ |
| $\{C, B\}$ | $\longleftrightarrow$ | $\{C, B, M\}$ | $\{R, B, G\}$ | $\longleftrightarrow$ | $\{R, B, G, M\}$ |
| $\{C, G\}$ | $\longleftrightarrow$ | $\{C, G, M\}$ | $\{C, R, B, G\}$ | $\longleftrightarrow$ | $\{C, R, B, G, M\}$ |

# Geometric Progressions

## Definition

- A sequence $a_1, a_2, \ldots, a_n$ is a **geometric progression** with a common positive ratio $q > 0$ if $a_i = qa_{i-1}$ for all $2 \leq i \leq n$

## Simplifying assumptions

- Set $a_1 = q$ and as a result the sequence becomes $q^1, q^2, \ldots, q^n$
- Add $a_0 = 1 = q^0$ to the beginning of the sequence and as a result the sequence becomes

$$q^0, q^1, q^2, \ldots, q^n$$

## Theorem

For a real number $q > 0$ and $q \neq 1$

$$\sum_{i=0}^{n-1} q^i = 1 + q + \cdots + q^{n-2} + q^{n-1} = \frac{q^n - 1}{q - 1}$$

# Proof By Induction

### Notations

- $L(n) = 1 + q + \cdots + q^{n-2} + q^{n-1}$
- $R(n) = \frac{q^n - 1}{q - 1}$

### The Induction base: $n = 1$

- $L(1) = R(1)$, because $L(1) = 1$ and $R(1) = \frac{q^1 - 1}{q - 1} = 1$

### The induction hypothesis: $L(k) = R(k)$ for $k \geq 1$

$$\sum_{i=0}^{k-1} q^i = 1 + q + \cdots + q^{k-2} + q^{k-1} = \frac{q^k - 1}{q - 1}$$

# Proof By Induction

**The inductive step:** $L(k+1) = R(k+1)$ **for** $k \geq 1$

$$
\begin{aligned}
L(k+1) &= 1 + q + \cdots + q^{k-1} + q^k \\
&= L(k) + q^k \\
&= R(k) + q^k \\
&= \frac{q^k - 1}{q - 1} + q^k \\
&= \frac{(q^k - 1) + ((q-1)q^k)}{q - 1} \\
&= \frac{(q^k - 1) + (q^{k+1} - q^k)}{q - 1} \\
&= \frac{q^{k+1} - 1}{q - 1} \\
&= R(k+1)
\end{aligned}
$$

# Another proof

**Theorem**

For a real number $q > 0$ and $q \neq 1$

$$\sum_{i=0}^{n-1} q^i = 1 + q + \cdots + q^{n-2} + q^{n-1} = \frac{q^n - 1}{q - 1}$$

**Proof**

$$\begin{aligned}(q-1)\sum_{i=0}^{n-1} q^i &= q\sum_{i=0}^{n-1} q^i - \sum_{i=0}^{n-1} q^i \\ &= (q + q^2 + \cdots + q^{n-1} + q^n) - (1 + q + \cdots + q^{n-2} + q^{n-1}) \\ &= q^n - 1\end{aligned}$$

# Geometric Progressions

## Corollary

For a real number $q > 0$ and $q \neq 1$

$$\sum_{i=1}^{n-1} q^i = q + \cdots + q^{n-2} + q^{n-1} = \frac{q^n - q}{q - 1}$$

## Proof

$$\begin{aligned}\sum_{i=1}^{n-1} q^i &= \sum_{i=0}^{n-1} q^i - 1 \\ &= \frac{q^n - 1}{q - 1} - \frac{q - 1}{q - 1} \\ &= \frac{q^n - q}{q - 1}\end{aligned}$$

# Geometric Progressions with $q = 2$

## Identity

$$\sum_{i=0}^{n-1} 2^i = 1 + 2 + 4 + \cdots + 2^{n-1} = \frac{2^n - 1}{2 - 1} = 2^n - 1$$

## Small $n$

$$\begin{aligned}
1 = 1 &= 2^1 - 1 \\
1 + 2 = 3 &= 2^2 - 1 \\
1 + 2 + 4 = 7 &= 2^3 - 1 \\
1 + 2 + 4 + 8 = 15 &= 2^4 - 1 \\
1 + 2 + 4 + 8 + 16 = 31 &= 2^5 - 1 \\
1 + 2 + 4 + 8 + 16 + 32 = 63 &= 2^6 - 1
\end{aligned}$$

# Geometric Progressions with $q = 3$

### Identity

$$\sum_{i=0}^{n-1} 3^i = 1 + 3 + 9 + \cdots + 3^{n-1} = \frac{3^n - 1}{3 - 1} = \frac{3^n - 1}{2}$$

### Small $n$

$$1 = 1 = \frac{3^1 - 1}{2} = \frac{3 - 1}{2}$$

$$1 + 3 = 4 = \frac{3^2 - 1}{2} = \frac{9 - 1}{2}$$

$$1 + 3 + 9 = 13 = \frac{3^3 - 1}{2} = \frac{27 - 1}{2}$$

$$1 + 3 + 9 + 27 = 40 = \frac{3^4 - 1}{2} = \frac{81 - 1}{2}$$

$$1 + 3 + 9 + 27 + 81 = 121 = \frac{3^5 - 1}{2} = \frac{243 - 1}{2}$$

# Geometric Progressions Visual Proofs

## $q = 3$

- https://www.youtube.com/watch?v=9IAm75UY2U8

## $q = 4$

- https://www.youtube.com/watch?v=yTpzDEDP090&list=PLZh9gzIvXQUsgw8W5TUVDtF0q4jEJ3iaw

## $q =$ 7 and all integers larger than 3

- https://www.youtube.com/watch?v=1wIdJxSfUz4&list=PLZh9gzIvXQUsgw8W5TUVDtF0q4jEJ3iaw

## $q = 8$

- https://www.youtube.com/watch?v=vcO5pa7iZOU

## $q = 9$

- https://www.youtube.com/watch?v=Ch7GFdsc9pQ

# Geometric Progressions for Large $q$

## First approximation: large $q$

$$\sum_{i=0}^{n-1} q^i = \frac{q^n - 1}{q - 1} = \frac{q^n}{q-1} - \frac{1}{q-1} \approx \frac{q^n}{q-1}$$

## Second approximation: very large $q$

$$\sum_{i=0}^{n-1} q^i \approx \frac{q^n}{q-1} \approx \frac{q^n}{q} = q^{n-1}$$

# Another Version of the Identity for the Sum

**Theorem**

For a real number $q > 0$ and $q \neq 1$

$$\sum_{i=0}^{n-1} q^i = 1 + q + \cdots + q^{n-1} = \frac{1-q^n}{1-q}$$

**Proof**

$$\begin{aligned}
\sum_{i=0}^{n-1} q^i &= \frac{q^n - 1}{q - 1} \\
&= \frac{(-1)(q^n - 1)}{(-1)(q - 1)} \\
&= \frac{1 - q^n}{1 - q}
\end{aligned}$$

# Another Version of the Identity for the Sum

## Corollary

For a real number $q > 0$ and $q \neq 1$

$$\sum_{i=1}^{n-1} q^i = q + q^2 + \cdots + q^{n-1} = \frac{q - q^n}{1 - q}$$

## Proof

$$\begin{aligned} \sum_{i=1}^{n-1} q^i &= \frac{q^n - q}{q - 1} \\ &= \frac{(-1)(q^n - q)}{(-1)(q - 1)} \\ &= \frac{q - q^n}{1 - q} \end{aligned}$$

# Which Identity To Use?

## The two identities

$$\sum_{i=0}^{n-1} q^i = 1 + q + \cdots + q^{n-1} = \frac{q^n - 1}{q - 1} \quad (1)$$

$$\sum_{i=0}^{n-1} q^i = 1 + q + \cdots + q^{n-1} = \frac{1 - q^n}{1 - q} \quad (2)$$

## Avoid negative numbers

- Use the first when $q > 1$ so both the numerator and the denominator are positive
- Use the second when $q < 1$ so both the numerator and the denominator are positive

# Geometric Progressions with $q = \frac{1}{2}$

## Identity

$$\sum_{i=0}^{n-1} \left(\frac{1}{2}\right)^i = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}}$$
$$= \frac{1 - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}}$$
$$= 2\left(1 - \left(\frac{1}{2}\right)^n\right) = 2 - \frac{1}{2^{n-1}}$$

## Small $n$

$$1 = 2 - \frac{1}{1} = 1$$
$$1 + \frac{1}{2} = 2 - \frac{1}{2} = \frac{3}{2}$$
$$1 + \frac{1}{2} + \frac{1}{4} = 2 - \frac{1}{4} = \frac{7}{4}$$
$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 2 - \frac{1}{8} = \frac{15}{8}$$

# Geometric Progressions with $q = \frac{2}{3}$

## Identity

$$
\begin{aligned}
\sum_{i=0}^{n-1} \left(\frac{2}{3}\right)^i &= 1 + \frac{2}{3} + \frac{4}{9} + \cdots + \frac{2^{n-1}}{3^{n-1}} \\
&= \frac{1 - \left(\frac{2}{3}\right)^n}{1 - \frac{2}{3}} \\
&= 3\left(1 - \left(\frac{2}{3}\right)^n\right) \\
&= 3 - \frac{2^n}{3^{n-1}}
\end{aligned}
$$

# Geometric Progressions with $q = \frac{k-1}{k}$

## Identity

$$\begin{aligned}
\sum_{i=0}^{n-1} \left(\frac{k-1}{k}\right)^i &= 1 + \frac{k-1}{k} + \frac{(k-1)^2}{k^2} + \cdots + \frac{(k-1)^{n-1}}{k^{n-1}} \\
&= \frac{1 - \left(\frac{k-1}{k}\right)^n}{1 - \frac{k-1}{k}} \\
&= k\left(1 - \left(\frac{k-1}{k}\right)^n\right) \\
&= k - \frac{(k-1)^n}{k^{n-1}}
\end{aligned}$$

# Geometric Progressions with $q = \frac{1}{2}$

## Identity

$$\begin{aligned}
\sum_{i=1}^{n-1} \left(\frac{1}{2}\right)^i &= \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}} \\
&= \frac{\frac{1}{2} - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}} \\
&= 2\left(\frac{1}{2} - \left(\frac{1}{2}\right)^n\right) = 1 - \frac{1}{2^{n-1}}
\end{aligned}$$

## Small numbers

$$\begin{aligned}
\frac{1}{2} &= 1 - \frac{1}{2} = \frac{1}{2} \\
\frac{1}{2} + \frac{1}{4} &= 1 - \frac{1}{4} = \frac{3}{4} \\
\frac{1}{2} + \frac{1}{4} + \frac{1}{8} &= 1 - \frac{1}{8} = \frac{7}{8} \\
\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} &= 1 - \frac{1}{16} = \frac{15}{16}
\end{aligned}$$

# Geometric Progressions with $q = \frac{1}{3}$

## Identity

$$\begin{aligned}
\sum_{i=1}^{n-1}\left(\frac{1}{3}\right)^i &= \frac{1}{3}+\frac{1}{9}+\cdots+\frac{1}{3^{n-1}} \\
&= \frac{\frac{1}{3}-\left(\frac{1}{3}\right)^n}{1-\frac{1}{3}} \\
&= \frac{3}{2}\left(\frac{1}{3}-\left(\frac{1}{3}\right)^n\right) \\
&= \frac{1}{2}-\frac{1}{3^{n-1}}
\end{aligned}$$

# Geometric Progressions with $q = \frac{1}{k}$

## Identity

$$\begin{aligned}
\sum_{i=1}^{n-1} \left(\frac{1}{k}\right)^i &= \frac{1}{k} + \frac{1}{k^2} + \cdots + \frac{1}{k^{n-1}} \\
&= \frac{\frac{1}{k} - \left(\frac{1}{k}\right)^n}{1 - \frac{1}{k}} \\
&= \frac{k}{k-1}\left(\frac{1}{k} - \left(\frac{1}{k}\right)^n\right) \\
&= \frac{1}{k-1} - \frac{1}{k^{n-1}}
\end{aligned}$$

# Infinite Geometric Progressions with $0 < q < 1$

## Theorem

$$\sum_{i=0}^{\infty} q^i = 1 + q + q^2 + \cdots = \frac{1}{1-q}$$

$$\sum_{i=1}^{\infty} q^i = q + q^2 + q^3 + \cdots = \frac{q}{1-q}$$

## Proof sketch

- $q^n \to 0$ when $n \to \infty$ and therefore $q^\infty = 0$

$$\sum_{i=0}^{\infty} q^i = \frac{1-q^\infty}{1-q} = \frac{1-0}{1-q} = \frac{1}{1-q}$$

$$\sum_{i=1}^{\infty} q^i = \frac{q-q^\infty}{1-q} = \frac{q-0}{1-q} = \frac{q}{1-q}$$

# Another proof

## Theorem

For a real number $0 < q < 1$

$$\sum_{i=0}^{\infty} q^i = 1 + q + q^2 + \cdots = \frac{1}{1-q}$$

## Proof

$$\begin{aligned}(1-q)\sum_{i=0}^{\infty} q^i &= \sum_{i=0}^{\infty} q^i - q\sum_{i=0}^{\infty} q^i \\ &= (1 + q + q^2 + \cdots) - (q + q^2 + q^3 + \cdots) \\ &= 1\end{aligned}$$

## Application

- https://www.youtube.com/watch?v=3cNdM7W0VlQ

# Infinite Geometric Progressions with $q = \frac{k-1}{k}$

## Small $k$

- $\sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{1}{1-\frac{1}{2}} = 2$
- $\sum_{i=0}^{\infty} \left(\frac{2}{3}\right)^i = 1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \cdots = \frac{1}{1-\frac{2}{3}} = 3$
- $\sum_{i=0}^{\infty} \left(\frac{3}{4}\right)^i = 1 + \frac{3}{4} + \frac{9}{16} + \frac{27}{64} + \cdots = \frac{1}{1-\frac{3}{4}} = 4$

## The general case

$$\sum_{i=0}^{\infty} \left(\frac{k-1}{k}\right)^i = 1 + \frac{k-1}{k} + \frac{(k-1)^2}{k^2} + \frac{(k-1)^3}{k^3} + \cdots$$

$$= \frac{1}{1 - \frac{k-1}{k}} = \frac{1}{\frac{1}{k}} = k$$

# Infinite Geometric Progressions with $q = \frac{1}{k}$

## Small $k$

- $\sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^i = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$
- $\sum_{i=1}^{\infty} \left(\frac{1}{3}\right)^i = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{1}{2}$
- $\sum_{i=1}^{\infty} \left(\frac{1}{4}\right)^i = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots = \frac{\frac{1}{4}}{1-\frac{1}{4}} = \frac{1}{3}$
- $\sum_{i=1}^{\infty} \left(\frac{1}{5}\right)^i = \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \cdots = \frac{\frac{1}{5}}{1-\frac{1}{5}} = \frac{1}{4}$
- $\sum_{i=1}^{\infty} \left(\frac{1}{6}\right)^i = \frac{1}{6} + \frac{1}{36} + \frac{1}{216} + \cdots = \frac{\frac{1}{6}}{1-\frac{1}{6}} = \frac{1}{5}$

## The general case

- $\sum_{i=1}^{\infty} \left(\frac{1}{k}\right)^i = \frac{1}{k} + \frac{1}{k^2} + \frac{1}{k^3} + \cdots = \frac{\frac{1}{k}}{1-\frac{1}{k}} = \frac{1}{k-1}$

# Infinite Geometric Progressions with $q = \frac{1}{k}$

## Visual Proofs

- $q = 1/2$: https://www.youtube.com/watch?v=saJJGNlsfn8
- $q = 1/3$:
  - https://www.youtube.com/watch?v=vfEDDI3vfHU
  - https://www.youtube.com/watch?v=RmTZmNrkqss
- $q = 1/4$: https://www.youtube.com/watch?v=8i1xj5ORwUw
- $q = 1/5$:
  - https://www.youtube.com/watch?v=yp7afEXYeC4
  - https://www.youtube.com/watch?v=IguRXWNwrn8&t=47s
- $q = 1/7$: https://www.youtube.com/watch?v=6wgCoIzsaA8
- $q = 1/9$: https://www.youtube.com/watch?v=C4t_ps3VKvI
- $q = 1/2, 1/3, \ldots, 1/9$: https://www.youtube.com/watch?v=JteQEN1XPyc

# Infinite Geometric Progressions with $q = \frac{k}{2k+1}$

## Small $k$

- $\sum_{i=1}^{\infty}\left(\frac{1}{3}\right)^i = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{1}{2}$
- $\sum_{i=1}^{\infty}\left(\frac{2}{5}\right)^i = \frac{2}{5} + \frac{4}{25} + \frac{8}{125} + \cdots = \frac{\frac{2}{5}}{1-\frac{2}{5}} = \frac{2}{3}$
- $\sum_{i=1}^{\infty}\left(\frac{3}{7}\right)^i = \frac{3}{7} + \frac{9}{49} + \frac{27}{243} + \cdots = \frac{\frac{3}{7}}{1-\frac{3}{7}} = \frac{3}{4}$

## The general case

- $\sum_{i=1}^{\infty}\left(\frac{k}{2k+1}\right)^i = \frac{k}{2k+1} + \frac{k^2}{(2k+1)^2} + \frac{k^3}{(2k+1)^3} + \cdots = \frac{\frac{k}{2k+1}}{1-\frac{k}{2k+1}} = \frac{k}{k+1}$

## A visual proof

- $q = \frac{4}{9}$: `https://www.youtube.com/watch?v=woKVh51KPl4`

# Sum of Powers of First $n$ Integers

## Small exponents

$$
\begin{array}{lclcll}
\sum_{i=1}^{n} i^0 & = & 1+1+\cdots+1 & = & n & \approx \frac{1}{1} n^1 \\
\sum_{i=1}^{n} i^1 & = & 1+2+\cdots+n & = & \frac{n(n+1)}{2} & \approx \frac{1}{2} n^2 \\
\sum_{i=1}^{n} i^2 & = & 1+4+9+\cdots+n^2 & = & \frac{n(n+1)(2n+1)}{6} & \approx \frac{1}{3} n^3 \\
\sum_{i=1}^{n} i^3 & = & 1+8+27+\cdots+n^3 & = & \frac{n^2(n+1)^2}{4} & \approx \frac{1}{4} n^4 \\
 & \vdots & & & & \vdots \\
\sum_{i=1}^{n} i^k & = & 1^k+2^k+\cdots+n^k & & & \approx \frac{1}{k+1} n^{k+1}
\end{array}
$$

# Sum of First $n$ Squares

## Identity

$$\sum_{i=1}^{n} i^2 = 1 + 4 + 9 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

## Correctness for Small $n$

$$\begin{array}{lcrcccc}
1 & = & 1 & = & \frac{1\cdot 2\cdot 3}{6} & = & \frac{6}{6} \\
1+4 & = & 5 & = & \frac{2\cdot 3\cdot 5}{6} & = & \frac{30}{6} \\
1+4+9 & = & 14 & = & \frac{3\cdot 4\cdot 7}{6} & = & \frac{84}{6} \\
1+4+9+16 & = & 30 & = & \frac{4\cdot 5\cdot 9}{6} & = & \frac{180}{6} \\
1+4+9+16+25 & = & 55 & = & \frac{5\cdot 6\cdot 11}{6} & = & \frac{330}{6} \\
1+4+9+16+25+36 & = & 91 & = & \frac{6\cdot 7\cdot 13}{6} & = & \frac{546}{6} \\
1+4+9+16+25+36+49 & = & 140 & = & \frac{7\cdot 8\cdot 15}{6} & = & \frac{840}{6}
\end{array}$$

# Sum of First $n$ Squares

## Visual proofs

- Proof 1: https://www.youtube.com/watch?v=-tJhH_k2LaM
- Proof 2: https://www.youtube.com/watch?v=UqVmocdLFGc
- Proof 3: https://www.youtube.com/watch?v=WidzHiUFWNA
- Proof 4: https://www.youtube.com/watch?v=a8j3sBrXchg
- Proof 5: https://www.youtube.com/watch?v=VYaEGvClg7Q

## Proof by induction

- https://www.youtube.com/watch?v=OI-nSvpZTpE

## Another identity with a double summation

- $\sum_{i=1}^{n} i^2 = \sum_{i=1}^{n} \sum_{j=i}^{n} j$
- A visual proof: https://www.youtube.com/watch?v=Q-frL0Ot2m4

# Sum of First $n$ Squares: Proof By Induction

## Notations

- $L(n) = 1 + 4 + 9 + \cdots + (n-1)^2 + n^2$
- $R(n) = \frac{n(n+1)(2n+1)}{6}$

## The induction base: $n = 1$

- $L(1) = R(1)$, because $L(1) = 1^2 = 1$ and $R(1) = \frac{1 \cdot 2 \cdot 3}{6} = 1$

## The induction hypothesis: $L(k) = R(k)$ for $k \geq 1$

$$\sum_{i=1}^{k} i^2 = 1 + 4 + 9 + \cdots + (k-1)^2 + k^2 = \frac{k(k+1)(2k+1)}{6}$$

# Sum of First $n$ Squares: Proof By Induction

**The inductive step:** $L(k+1) = R(k+1)$ **for** $k \geq 1$

$$\begin{aligned}
L(k+1) &= 1 + 4 + 9 + \cdots + k^2 + (k+1)^2 \\
&= L(k) + (k+1)^2 \\
&= R(k) + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= \frac{(2k^3 + 3k^2 + k) + (6k^2 + 12k + 6)}{6} \\
&= \frac{2k^3 + 9k^2 + 13k + 6}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6} \\
&= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} \\
&= R(k+1)
\end{aligned}$$

# Sum of First $n$ Squares: Proof By Induction

**The inductive step:** $L(k+1) = R(k+1)$ **for** $k \geq 1$

$$
\begin{aligned}
L(k+1) &= 1 + 4 + 9 + \cdots + k^2 + (k+1)^2 \\
&= L(k) + (k+1)^2 \\
&= R(k) + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= \frac{(k^2+k)(2k+1)}{6} + \frac{6(k+1)^2}{6} \\
&= \frac{(2k^3+3k^2+k)+(6k^2+12k+6)}{6} \\
&= \frac{2k^3+9k^2+13k+6}{6}
\end{aligned}
$$

$$
\begin{aligned}
R(k+1) &= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6} \\
&= \frac{(k^2+3k+2)(2k+3)}{6} \\
&= \frac{2k^3+9k^2+13k+6}{6}
\end{aligned}
$$

# Sum of First $n$ Cubes

## Identity

$$\begin{aligned}\sum_{i=1}^{n} i^3 &= 1+8+27+\cdots+(n-1)^3+n^3 \\ &= \frac{n^2(n+1)^2}{4} \\ &= \left(\frac{n(n+1)}{2}\right)^2 \\ &= (1+2+3+\cdots+(n-1)+n)^2\end{aligned}$$

## Visual Proofs

- Proof 1: https://www.youtube.com/watch?v=YQLicI8R4Gs
- Proof 2: https://www.youtube.com/watch?v=Ye9OPNqV9FA
- Proof 3: https://www.youtube.com/watch?v=NxOcT_VKQR0
- Proof 4: https://www.youtube.com/watch?v=jWpyrXYZNiI
- Proof 5: https://www.youtube.com/watch?v=d1yM6Rq7Tfw

# Sum of First $n$ Cubes

## Correctness for Small $n$

$$\begin{aligned}
1 &= 1 = \frac{1^2 \cdot 2^2}{4} = \frac{4}{4} \\
1+8 &= 9 = \frac{2^2 \cdot 3^2}{4} = \frac{36}{4} \\
1+8+27 &= 36 = \frac{3^2 \cdot 4^2}{4} = \frac{144}{4} \\
1+8+27+64 &= 100 = \frac{4^2 \cdot 5^2}{4} = \frac{400}{4} \\
1+8+27+64+125 &= 225 = \frac{5^2 \cdot 6^2}{4} = \frac{900}{4} \\
1+8+27+64+125+216 &= 441 = \frac{6^2 \cdot 7^2}{4} = \frac{1764}{4} \\
1+8+27+64+125+216+343 &= 784 = \frac{7^2 \cdot 8^2}{4} = \frac{3136}{4}
\end{aligned}$$

# Sum of First $n$ Cubes: Proof By Induction

**Notations**

- $L(n) = 1 + 8 + 27 + \cdots + (n-1)^3 + n^3$
- $R(n) = \frac{n^2(n+1)^2}{4}$

**The induction base:** $n = 1$

- $L(1) = R(1)$, because $L(1) = 1^3 = 1$ and $R(1) = \frac{1^2 \cdot 2^2}{4} = 1$

**The induction hypothesis:** $L(k) = R(k)$ **for** $k \geq 1$

$$\sum_{i=1}^{k} i^3 = 1 + 8 + 27 + \cdots + (k-1)^3 + k^3 = \frac{k^2(k+1)^2}{4}$$

# Sum of First $n$ Cubes: Proof By Induction

**The inductive step:** $L(k+1) = R(k+1)$ **for** $k \geq 1$

$$\begin{aligned}
L(k+1) &= 1 + 8 + 27 + \cdots + k^3 + (k+1)^3 \\
&= L(k) + (k+1)^3 \\
&= R(k) + (k+1)^3 \\
&= \frac{k^2(k+1)^2}{4} + (k+1)^3 \\
&= \frac{k^2(k+1)^2 + 4(k+1)^3}{4} \\
&= \frac{(k+1)^2(k^2+4k+4)}{4} \\
&= \frac{(k+1)^2(k+2)^2}{4} \\
&= R(k+1)
\end{aligned}$$

# Sum Of Fractions Identity

## Identity

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n}$$

## Correctness for Small *n*

$$\begin{aligned}
\tfrac{1}{2} &= \tfrac{1}{2} &&= 1 - \tfrac{1}{2} \\
\tfrac{1}{3} + \tfrac{1}{4} &= \tfrac{7}{12} &&= 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} \\
\tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} &= \tfrac{37}{60} &&= 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} \\
\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8} &= \tfrac{533}{840} &&= 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \tfrac{1}{7} - \tfrac{1}{8}
\end{aligned}$$

# Sum Of Fractions Identity: Proof By Induction

## Notations

- $L(n) = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$
- $R(n) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n}$

## The induction base: $n = 1$

- $L(1) = R(1)$, because $L(1) = \frac{1}{2}$ and $R(1) = 1 - \frac{1}{2} = \frac{1}{2}$

## The induction hypothesis: $L(k) = R(k)$ for $k \geq 1$

$$\frac{1}{k+1} + \frac{1}{k+2} + \cdots + \frac{1}{2k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2k-1} - \frac{1}{2k}$$

# Sum Of Fractions Identity: Proof By Induction

**The inductive step:** $L(k+1) = R(k+1)$ **for** $k \geq 1$

$$
\begin{aligned}
L(k+1) &= \frac{1}{k+2} + \frac{1}{k+3} + \cdots + \frac{1}{2k} + \frac{1}{2k+1} + \frac{1}{2k+2} \\
&= \frac{1}{k+1} + \frac{1}{k+2} + \cdots + \frac{1}{2k} + \frac{1}{2k+1} + \frac{1}{2k+2} - \frac{1}{k+1} \\
&= L(k) + \frac{1}{2k+1} + \left(\frac{1}{2k+2} - \frac{1}{k+1}\right) \\
&= L(k) + \frac{1}{2k+1} - \frac{1}{2k+2} \\
&= R(k) + \frac{1}{2k+1} - \frac{1}{2k+2} \\
&= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2k-1} - \frac{1}{2k} + \frac{1}{2k+1} - \frac{1}{2k+2} \\
&= R(k+1)
\end{aligned}
$$

# Strong Induction

## The strong version of the principle

- **Setting:** Let $P_n$ be a statement about all positive integers $n = 1, 2, 3, \ldots$
- **Induction base:** $P_1, \ldots, P_m$ are true for some $m \geq 1$
- **Induction hypothesis:** $P_1, P_2, \ldots, P_k$ are true for some $k \geq m$
- **Inductive step:** $P_{k+1}$ is implied by a non-empty subset of statements from the set $\{P_1, P_2, \ldots, P_k\}$

## Online resource

- Strong induction: prime factorization and another example
  `https://www.youtube.com/watch?v=g9YSizeBwqo&t=317s`

# Prime Factorization

## Theorem

- Every positive integer $n \geq 2$ is a power of a prime number or the product of powers of prime numbers

## Proof by Induction

- **Induction base:** $2 = 2^1$ is a power of a prime
- **Induction hypothesis:** Assume every positive integer less than $n$ is a prime number or a product of powers of prime numbers
- **Inductive step:**
  - If $n$ is a prime, then $n = n^1$ is a power of a prime
  - Otherwise, $n = m \cdot h$ is a product of two numbers $m < n$ and $h < n$
  - By the induction hypothesis, both $m$ and $h$ are power of prime numbers or products of prime numbers
  - Therefore, $n = m \cdot h$ is also a power of a prime number or a product of powers of prime numbers

# Prime Factorization

## Example I

- $90 = 15 \cdot 6 = (3 \cdot 5)(2 \cdot 3)$
- Therefore by induction, $90 = 2 \cdot 3^2 \cdot 5$

## Example II

- $216 = 12 \cdot 18 = (2^2 \cdot 3)(2 \cdot 3^2)$
- Therefore by induction, $216 = 2^3 \cdot 3^3$

## Example III

- $128 = 8 \cdot 16 = 2^3 \cdot 2^4$
- Therefore by induction, $128 = 2^7$

# A Chocolate Bar Problem

## Problem

- A chocolate bar consisting of $n \geq 1$ unit squares is arranged as an $m \times h$ rectangular grid ($n = m \cdot h$)
- The goal is to split the bar into $n$ individual unit squares by breaking along the lines
- It is not allowed to break more than one rectangular piece at a time (e.g., by piling them together)
- What is the required number of breaks?

## Online resource

- `https://www.youtube.com/watch?v=yftf3fs9k6s`

# A Chocolate Bar Problem

## Claim

- For $0 \leq k \leq n-1$, after $k$ breaks, there are $k+1$ pieces

## Proof by induction sketch

- By induction on the number of breaks
- After $k = 0$ breaks there is one piece and indeed $1 = 0 + 1$
- Before the $k^{\text{th}}$ break, by the induction hypothesis, there were $k = (k-1) + 1$ pieces
- After the $k^{\text{th}}$ break there are $k+1$ pieces because the break replaces one of the pieces with two pieces

## Corollary

- After $n-1$ breaks there are $n$ pieces. That is, the bar was split into $n$ individuals unit squares

# More Examples

## Three problems

- L-shape tiles of size 3 can tile any square of size $2^n \times 2^n$ small squares with any missing square
- Number of steps needed to solve the tower of Hanoi problem
- Any partition of the circle with chords can be face-colored with two colors

## Online resource

- `https://www.youtube.com/watch?v=5Hn8vUE3cBQ`

## Remark

- All proofs imply an algorithm!

# A False Divisibility Claim

**Claim**

- $n^3 - n + 1$ is divisible by 3

**Wrong for small values of $n$**

$$\begin{aligned}
1^3 - 1 + 1 &= 1 = 3 \cdot 0 + 1 \\
2^3 - 2 + 1 &= 7 = 3 \cdot 2 + 1 \\
3^3 - 3 + 1 &= 25 = 3 \cdot 8 + 1 \\
4^3 - 4 + 1 &= 61 = 3 \cdot 20 + 1
\end{aligned}$$

# Proof By Induction

### The induction base

- Skip the base case!

### The induction hypothesis for $k$

- Assume that $k^3 - k + 1 = 3q$ is divisible by 3

### The inductive step for $k + 1$

$$
\begin{aligned}
(k+1)^3 - (k+1) + 1 &= k^3 + 3k^2 + 3k + 1 - k - 1 + 1 \\
&= (k^3 - k + 1) + (3k^2 + 3k) \\
&= 3q + 3(k^2 + k) \quad (\ast \text{ the induction hypothesis } \ast) \\
&= 3(q + k^2 + k) \quad (\ast \text{ Q.E.D. } \ast)
\end{aligned}
$$

# Correct Claim

**Theorem**

- $n^3 - n$ is divisible by 6

**Small values of $n$**

$$\begin{aligned} 1^3 - 1 &= 0 = 6 \cdot 0 \\ 2^3 - 2 &= 6 = 6 \cdot 1 \\ 3^3 - 3 &= 24 = 6 \cdot 4 \\ 4^3 - 4 &= 60 = 6 \cdot 10 \end{aligned}$$

**Proof**

- $n^3 - n = n(n^2 - 1) = (n-1)n(n+1)$
- That is, $n^3 - n$ is a product of three consecutive integers
- One of them must be divisible by 3
- One (could be the same integer) must be even
- Therefore, the product of the three integers must be divisible by 6

# $6n = 0$ for All Integers $n \geq 0$???

## Proof by induction for $n \geq 0$

- Clearly, if $n = 0$, then $6n = 0$
- Let $n > 0$ and assume that $6k = 0$ for all $0 \leq k < n$
- Let $n = h + m$ for integers $0 \leq h < n$ and $0 \leq m < n$
- By the **strong** induction hypothesis, $6h = 0$ and $6m = 0$.
- Therefore $6n = 6(h + m) = 6h + 6m = 0 + 0 = 0$
- Q.E.D.

## Where is the Error?

- The proof fails for $n = 1$
- 1 cannot be expressed as a sum of two non-negative integers that are smaller than 1

# All Horses in the World are of the Same Color

## Proof by induction on the number of horses

- The base of the induction is that if there is one horse, then it is trivially the same color as itself
- Suppose that there are $n$ horses, numbered 1 through $n$
- By the induction hypothesis, the $n-1$ horses 1 through $n-1$ are all of the same color
- Assume this color is black. In particular, horse 2 is black
- This means that the $n-1$ horses 2 through $n$ must be black by the induction hypothesis
- Therefore, all of the horses 1 through $n$ are of the same color

## Where is the Error?

- Proof fails for $n = 2$ in which horse 2 may be of a different color

## Online resources

- `https://www.youtube.com/watch?v=sCUg5DNCETI`
- `https://en.wikipedia.org/wiki/All_horses_are_the_same_color`

# Why Induction Works?

## "Justification" with the Well-Ordering Principle

- Assume that there exists $j \geq 2$ such that $P_j$ is **false**
- Let $S$ be the set of **all** integers $h \geq 1$ for which $P_h$ is **false**
  - $*$ $S$ is a non empty set that can contain infinite number of integers
- Let $k + 1$ be the **minimum** integer in $S$
  - $*$ The **Well-Ordering Principle**
- $k \geq 1$ since by the **induction base** $P_1$ is true
- $P_k$ is true and $P_{k+1}$ is false by the minimality of $k + 1$
- A **contradiction** to the **inductive step**

# Notations

**The induction variable**

- The **inductive step** could be that $P_{n+1}$ is implied by $P_n$ and then $P_n$ is the **induction hypothesis**
- The **inductive step** could be that $P_n$ is implied by $P_{n-1}$ and then $P_{n-1}$ is the **induction hypothesis**