# Discrete Structures: Logic

Amotz Bar-Noy

Department of Computer and Information Science
Brooklyn College

# Boolean Functions With One Variables

## Definition

- Let $x$ be a boolean variable: $x \in \{$**TRUE,FALSE**$\}$.
- A boolean function $F(x) \in \{$**TRUE,FALSE**$\}$ is determined by its values for each one of the two possible assignments to $x$:
  - $*$ $x$ is **TRUE**
  - $*$ $x$ is **FALSE**

## Notation

- $\{\mathbf{T}, \mathbf{F}\}$ for $\{$**TRUE,FALSE**$\}$

## The four ($4 = 2^2$) functions

- The $\mathcal{IDENTITY}$ function: $F(x) = x$
- The $\mathcal{NOT}$ function: $F(x) = \neg x$
- The $\mathcal{TRUE}$ function: $F(x) =$ **TRUE**
- The $\mathcal{FALSE}$ function: $F(x) =$ **FALSE**

# Boolean Functions With Two Variables

## Definition

- Let $x$ and $y$ be two boolean variables: $x, y \in \{$**TRUE,FALSE**$\}$.
- A boolean function $F(x, y) \in \{$**TRUE,FALSE**$\}$ is determined by its values for each of the four possible assignments to $x$ and $y$:
    - Both $x$ and $y$ are **TRUE**
    - $x$ is **TRUE** and $y$ is **FALSE**
    - $x$ is **FALSE** and $y$ is **TRUE**
    - Both $x$ and $y$ are **FALSE**

## Number of two-variable functions

- There are $16 = 2^4$ possible boolean functions with two variables.

# Boolean Functions With $k$ Variables

## Counting the number of $k$-variable boolean functions

- Let $x_1, x_2, \ldots, x_k$ be $k$ boolean variables:
  - $(x_i \in \{$**TRUE,FALSE**$\})$ For all $1 \leq i \leq n$
- There are $2^k$ possible **TRUE** or **FALSE** assignments to the $k$ variables $x_1, x_2, \ldots, x_k$.
- A boolean function $F(x_1, \ldots, x_k) \in \{$**TRUE,FALSE**$\}$ is determined by its values for each one of these $2^k$ assignments.
- Therefore, there are $2^{2^k}$ different $k$-variable bollean functions.

## $2^{2^k}$ grows very fast!

- $4, 16, 256, 65536, 4294967296, 2^{64}, 2^{128} \ldots$
- There are more than 4 billions 5-variable boolean functions.

# Six "Trivial" Two Variable Functions

| $x$ | $y$ | $\mathcal{TRUE}$<br>$T$ | $\mathcal{FALSE}$<br>$F$ | $\mathcal{ID}(\mathcal{X})$<br>$x$ | $\mathcal{ID}(\mathcal{Y})$<br>$y$ | $\mathcal{NOT}(\mathcal{X})$<br>$\neg x$ | $\mathcal{NOT}(\mathcal{Y})$<br>$\neg y$ |
|---|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $F$ | $T$ | $T$ | $F$ | $F$ |
| $T$ | $F$ | $T$ | $F$ | $T$ | $F$ | $F$ | $T$ |
| $F$ | $T$ | $T$ | $F$ | $F$ | $T$ | $T$ | $F$ |
| $F$ | $F$ | $T$ | $F$ | $F$ | $F$ | $T$ | $T$ |

# Six "Useful" Or "Popular" Two Variable Functions

| $x$ | $y$ | $\mathcal{AND}$ $x \wedge y$ | $\mathcal{OR}$ $x \vee y$ | $\mathcal{NAND}$ $\neg(x \wedge y)$ | $\mathcal{NOR}$ $\neg(x \vee y)$ | $\mathcal{XOR}$ $x \oplus y$ | $\mathcal{EQUIV}$ $x \equiv y$ |
|---|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $F$ | $F$ | $F$ | $T$ |
| $T$ | $F$ | $F$ | $T$ | $T$ | $F$ | $T$ | $F$ |
| $F$ | $T$ | $F$ | $T$ | $T$ | $F$ | $T$ | $F$ |
| $F$ | $F$ | $F$ | $F$ | $T$ | $T$ | $F$ | $T$ |

**Representing $\mathcal{XOR}$ and $\mathcal{EQUIV}$ with $\mathcal{AND}$ and $\mathcal{OR}$**

- $\mathcal{XOR}$: $(x \vee y) \wedge (\neg x \vee \neg y)$
- $\mathcal{EQUIV}$: $(x \wedge y) \vee (\neg x \wedge \neg y)$

# Four Conditional Two Variable Functions

| $x$ | $y$ | $\mathcal{X}$–$\mathcal{IMPLIES}$–$\mathcal{Y}$ <br> $x \rightarrow y \equiv \neg x \vee y$ | $\mathcal{Y}$–$\mathcal{IMPLIES}$–$\mathcal{X}$ <br> $y \rightarrow x \equiv x \vee \neg y$ |
|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $T$ |
| $F$ | $T$ | $T$ | $F$ |
| $F$ | $F$ | $T$ | $T$ |

| $x$ | $y$ | $\mathcal{NOT}$($\mathcal{X}$–$\mathcal{IMPLIES}$–$\mathcal{Y}$) <br> $\neg(x \rightarrow y) \equiv x \wedge \neg y$ | $\mathcal{NOT}$($\mathcal{Y}$–$\mathcal{IMPLIES}$–$\mathcal{X}$) <br> $\neg(y \rightarrow x) \equiv \neg x \wedge y$ |
|---|---|---|---|
| $T$ | $T$ | $F$ | $F$ |
| $T$ | $F$ | $T$ | $F$ |
| $F$ | $T$ | $F$ | $T$ |
| $F$ | $F$ | $F$ | $F$ |

# All 16 Two Variables Functions

| Function$(x, y)$ | $(T, T)$ | $(T, F)$ | $(F, T)$ | $(F, F)$ |
|---|---|---|---|---|
| $\mathcal{TRUE}$ | $T$ | $T$ | $T$ | $T$ |
| $\mathcal{OR}$ | $T$ | $T$ | $T$ | $F$ |
| $\mathcal{Y}$–$\mathcal{IMPLIES}$–$\mathcal{X}$ | $T$ | $T$ | $F$ | $T$ |
| $\mathcal{ID}(\mathcal{X})$ | $T$ | $T$ | $F$ | $F$ |
| $\mathcal{X}$–$\mathcal{IMPLIES}$–$\mathcal{Y}$ | $T$ | $F$ | $T$ | $T$ |
| $\mathcal{ID}(\mathcal{Y})$ | $T$ | $F$ | $T$ | $F$ |
| $\mathcal{EQUIV}$ | $T$ | $F$ | $F$ | $T$ |
| $\mathcal{AND}$ | $T$ | $F$ | $F$ | $F$ |
| $\mathcal{NAND}$ | $F$ | $T$ | $T$ | $T$ |
| $\mathcal{XOR}$ | $F$ | $T$ | $T$ | $F$ |
| $\mathcal{NOT}(\mathcal{Y})$ | $F$ | $T$ | $F$ | $T$ |
| $\mathcal{NOT}(\mathcal{X}$–$\mathcal{IMPLIES}$–$\mathcal{Y})$ | $F$ | $T$ | $F$ | $F$ |
| $\mathcal{NOT}(\mathcal{X})$ | $F$ | $F$ | $T$ | $T$ |
| $\mathcal{NOT}(\mathcal{Y}$–$\mathcal{IMPLIES}$–$\mathcal{X})$ | $F$ | $F$ | $T$ | $F$ |
| $\mathcal{NOR}$ | $F$ | $F$ | $F$ | $T$ |
| $\mathcal{FALSE}$ | $F$ | $F$ | $F$ | $F$ |

# Laws of Logic: One Variable

**The identity laws:**

- $x \vee F \equiv x$
- $x \wedge T \equiv x$

**The domination laws:**

- $x \vee T \equiv T$
- $x \wedge F \equiv F$

**The idempotent laws:**

- $x \vee x \equiv x$
- $x \wedge x \equiv x$

**The complement laws:**

- $x \vee \neg x \equiv T$
- $x \wedge \neg x \equiv F$

**The double negation law:**

- $\neg\neg x \equiv x$

# Laws of Logic: Two and Three Variables

## The commutative Laws:

- $x \vee y \equiv y \vee x$
- $x \wedge y \equiv y \wedge x$

## The associative laws:

- $(x \vee y) \vee z \equiv x \vee (y \vee z) \equiv (x \vee y \vee z)$
- $(x \wedge y) \wedge z \equiv x \wedge (y \wedge z) \equiv (x \wedge y \wedge z)$

## The distributive laws:

- $x \vee (y \wedge z) \equiv (x \vee y) \wedge (x \vee z)$
- $x \wedge (y \vee z) \equiv (x \wedge y) \vee (x \wedge z)$

## The absorption laws:

- $x \vee (x \wedge y) \equiv x$
- $x \wedge (x \vee y) \equiv x$

# Boolean Formulas

## Definition

- A boolean **formula** (**expression**, **proposition**) is a sequence, with or without parentheses, of one or two variable boolean functions.
- The value of a boolean formula is **TRUE** ($T$) or **FALSE** ($F$) depending on the **TRUE/FALSE** assignments to all the boolean variables that appear in the formula.

## Evaluating a boolean formula

- A formula is processed in order from left to right.
- Negations are evaluted first.
- Priorities are given to parentheses.

# Evaluating Boolean Formulas: Examples

$(\mathbf{x} \vee \mathbf{y}) \wedge (\neg \mathbf{y} \wedge \mathbf{z})$

- The formula is **TRUE** only if both $(x \vee y)$ and $(\neg y \wedge z)$ are **TRUE**.
- $(\neg y \wedge z)$ is **TRUE** only if $y$ is **FALSE** and $z$ is **TRUE**.
- $x$ must be **TRUE** to force $(x \vee y)$ to be **TRUE** when $y$ is **FALSE**.
- In summary, the formula is **TRUE** only if $x$ is **TRUE**, $y$ is **FALSE**, and $z$ is **TRUE**.

$(\mathbf{x} \vee \mathbf{y}) \wedge (\neg \mathbf{y} \oplus \mathbf{z}) \wedge (\neg \mathbf{z} \equiv \mathbf{w})$

- When all the variables are **TRUE** then the formula is **FALSE** because the sub-formula $(\neg z \equiv w)$ is **FALSE**.
- When all the variables except $w$ are **TRUE** then the formula is **TRUE** because all the sub-formulas are **TRUE**.

# Truth Tables

## Definition

- A formula with $k$ variables has a value for each one of the $2^k$ possible **TRUE/FALSE** assignments to its $k$ variables.
- The **truth table** of a formula with $k$ variables is a table with $2^k$ rows which gives the value of the formula for each one of the $2^k$ assignments
- The first $k$ columns of the truth table represent the **TRUE/FALSE** assignments to the $k$ variables .
- The last column of the truth table represents the value of the formula.
- Usually there are additional columns that represent the values of sub-formulas helping verifying the correctness of the last column.

# Truth Table for: $(x \vee y) \wedge (\neg y \wedge z)$

| $x$ | $y$ | $z$ | $(x \vee y)$ | $\neg y$ | $(\neg y \wedge z)$ | formula |
|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $F$ | $F$ | $F$ |
| $T$ | $T$ | $F$ | $T$ | $F$ | $F$ | $F$ |
| $T$ | $F$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $T$ | $T$ | $F$ | $F$ |
| $F$ | $T$ | $T$ | $T$ | $F$ | $F$ | $F$ |
| $F$ | $T$ | $F$ | $T$ | $F$ | $F$ | $F$ |
| $F$ | $F$ | $T$ | $F$ | $T$ | $T$ | $F$ |
| $F$ | $F$ | $F$ | $F$ | $T$ | $F$ | $F$ |

# Truth Table for: $(x \vee y) \wedge (\neg y \oplus z) \wedge (\neg z \equiv w)$

| $x$ | $y$ | $z$ | $w$ | $(x \vee y)$ | $(\neg y \oplus z)$ | $(\neg z \equiv w)$ | formula |
|---|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $F$ | $F$ |
| $T$ | $T$ | $T$ | $F$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $T$ | $F$ | $T$ | $T$ | $F$ | $T$ | $F$ |
| $T$ | $T$ | $F$ | $F$ | $T$ | $F$ | $F$ | $F$ |
| $T$ | $F$ | $T$ | $T$ | $T$ | $F$ | $F$ | $F$ |
| $T$ | $F$ | $T$ | $F$ | $T$ | $F$ | $T$ | $F$ |
| $T$ | $F$ | $F$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $F$ | $T$ | $T$ | $F$ | $F$ |
| $F$ | $T$ | $T$ | $T$ | $T$ | $T$ | $F$ | $F$ |
| $F$ | $T$ | $T$ | $F$ | $T$ | $T$ | $T$ | $T$ |
| $F$ | $T$ | $F$ | $T$ | $T$ | $F$ | $T$ | $F$ |
| $F$ | $T$ | $F$ | $F$ | $T$ | $F$ | $F$ | $F$ |
| $F$ | $F$ | $T$ | $T$ | $F$ | $F$ | $F$ | $F$ |
| $F$ | $F$ | $T$ | $F$ | $F$ | $F$ | $T$ | $F$ |
| $F$ | $F$ | $F$ | $T$ | $F$ | $T$ | $T$ | $F$ |
| $F$ | $F$ | $F$ | $F$ | $F$ | $T$ | $F$ | $F$ |

# Satisfiability

## Definition

- A boolean formula is **satisfied** if there exists at least one **TRUE/FALSE** assignment to its variables for which the value of the formula is **TRUE**.
- Such assignments are the **truth assignments** of the formula.

**Example:** $(x \vee y) \wedge (\neg y \wedge z)$

- This formula is satisfied and it has only one truth assignments:
  - $x = T$, $y = F$, and $z = T$

**Example:** $(x \vee y) \wedge (\neg y \oplus z) \wedge (\neg z \equiv w)$

- This formula is satisfied and has three truth assignments:
  - $x = T$, $y = T$, $z = T$ and $w = F$
  - $x = T$, $y = F$, $z = F$ and $w = T$
  - $x = F$, $y = T$, $z = T$ and $w = F$

# Tautologies and Contradictions

## Definitions

- A boolean formula is a **tautology** if its value is **TRUE** for any **TRUE/FALSE** assignment to its variables.
- A boolean formula is a **contradiction** if its value is **FALSE** for any **TRUE/FALSE** assignment to its variables.

## Trivial examples

- $(x \vee \neg x)$ is a tautology.
- $(x \wedge \neg x)$ is a contradiction.

## Observations

- A tautology is satisfied by all possible assignments to its variables and the entries in the last column of its truth table are all $T$.
- A contradiction is a non-satisfied formula and the entries in the last column of its truth table are all $F$.

# A Tautology Example

### Theorem

- $P \equiv x \vee \neg(x \wedge y)$ is a tautology.

### Proof

- If $x$ is **TRUE** then $P$ is **TRUE**.
- If $x$ is **FALSE** then $x \wedge y$ is **FALSE** implying that $\neg(x \wedge y)$ is **TRUE** implying that $P$ is **TRUE**.
- It follows that $P$ is always **TRUE**.

### Truth table

| $x$ | $y$ | $x \wedge y$ | $\neg(x \wedge y)$ | $x \vee \neg(x \wedge y)$ |
|---|---|---|---|---|
| $T$ | $T$ | $T$ | $F$ | $T$ |
| $T$ | $F$ | $F$ | $T$ | $T$ |
| $F$ | $T$ | $F$ | $T$ | $T$ |
| $F$ | $F$ | $F$ | $T$ | $T$ |

# A Contradiction Example

## Theorem

- $P \equiv x \wedge \neg(x \vee y)$ is a contradiction.

## Proof

- If $x$ is **FALSE** then $P$ is **FALSE**.
- If $x$ is **TRUE** then $x \vee y$ is **TRUE** implying that $\neg(x \vee y)$ is **FALSE** implying that $P$ is **FALSE**.
- It follows that $P$ is always **FALSE**.

## Truth table

| $x$ | $y$ | $x \vee y$ | $\neg(x \vee y)$ | $x \wedge \neg(x \vee y)$ |
|---|---|---|---|---|
| $T$ | $T$ | $T$ | $F$ | $F$ |
| $T$ | $F$ | $T$ | $F$ | $F$ |
| $F$ | $T$ | $T$ | $F$ | $F$ |
| $F$ | $F$ | $F$ | $T$ | $F$ |

# Proving the Associative Laws Using Truth Tables

## The $\mathcal{AND}$ associative law

- $(x \wedge y) \wedge z \equiv x \wedge (y \wedge z)$

## Proof

- The last columns in both truth tables are identical:

| $x$ | $y$ | $z$ | $x \wedge y$ | $(x \wedge y) \wedge z$ |
|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $T$ | $F$ | $T$ | $F$ |
| $T$ | $F$ | $T$ | $F$ | $F$ |
| $T$ | $F$ | $F$ | $F$ | $F$ |
| $F$ | $T$ | $T$ | $F$ | $F$ |
| $F$ | $T$ | $F$ | $F$ | $F$ |
| $F$ | $F$ | $T$ | $F$ | $F$ |
| $F$ | $F$ | $F$ | $F$ | $F$ |

| $x$ | $y$ | $z$ | $y \wedge z$ | $x \wedge (y \wedge z)$ |
|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $T$ | $F$ | $F$ | $F$ |
| $T$ | $F$ | $T$ | $F$ | $F$ |
| $T$ | $F$ | $F$ | $F$ | $F$ |
| $F$ | $T$ | $T$ | $T$ | $F$ |
| $F$ | $T$ | $F$ | $F$ | $F$ |
| $F$ | $F$ | $T$ | $F$ | $F$ |
| $F$ | $F$ | $F$ | $F$ | $F$ |

# Proving the Associative Laws Using Truth Tables

## The $\mathcal{OR}$ associative law

- $(x \vee y) \vee z \equiv x \vee (y \vee z)$

## Proof

- The last columns in both truth tables are identical:

| $x$ | $y$ | $z$ | $x \vee y$ | $(x \vee y) \vee z$ |
|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $T$ | $F$ | $T$ | $T$ |
| $T$ | $F$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $T$ | $T$ |
| $F$ | $T$ | $T$ | $T$ | $T$ |
| $F$ | $T$ | $F$ | $T$ | $T$ |
| $F$ | $F$ | $T$ | $F$ | $T$ |
| $F$ | $F$ | $F$ | $F$ | $F$ |

| $x$ | $y$ | $z$ | $y \vee z$ | $x \vee (y \vee z)$ |
|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $T$ | $F$ | $T$ | $T$ |
| $T$ | $F$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $F$ | $T$ |
| $F$ | $T$ | $T$ | $T$ | $T$ |
| $F$ | $T$ | $F$ | $T$ | $T$ |
| $F$ | $F$ | $T$ | $T$ | $T$ |
| $F$ | $F$ | $F$ | $F$ | $F$ |

# The De Morgan's Laws

**The two variables case**

$$\begin{aligned}\neg(x \wedge y) &\equiv \neg x \vee \neg y \\ \neg(x \vee y) &\equiv \neg x \wedge \neg y\end{aligned}$$

**The three variables case**

$$\begin{aligned}\neg(x \wedge y \wedge z) &\equiv (\neg x) \vee (\neg y) \vee (\neg z) \\ \neg(x \vee y \vee z) &\equiv (\neg x) \wedge (\neg y) \wedge (\neg z)\end{aligned}$$

**The four variables case**

$$\begin{aligned}\neg(x \wedge y \wedge z \wedge w) &\equiv (\neg x) \vee (\neg y) \vee (\neg z) \vee (\neg w) \\ \neg(x \vee y \vee z \vee w) &\equiv (\neg x) \wedge (\neg y) \wedge (\neg z) \wedge (\neg w)\end{aligned}$$

# The De Morgan's Laws

## A "binary" world

- The outside is **rainy** (**R**) or **sunny** (**S**).
- The outside is **cold** (**C**) or **hot** (**H**).

## Person I

- Alice does not leave the house if it is (**rainy and cold**) outside.
- Alice left the house: $\neg(\mathbf{R} \wedge \mathbf{C}) \equiv (\neg\mathbf{R} \vee \neg\mathbf{C}) \equiv (\mathbf{S} \vee \mathbf{H})$.
- It is **sunny or hot** outside.

## Person II

- Bob does not leave the house if it is (**rainy or cold**) outside.
- Bob left the house: $\neg(\mathbf{R} \vee \mathbf{C}) \equiv (\neg\mathbf{R} \wedge \neg\mathbf{C}) \equiv (\mathbf{S} \wedge \mathbf{H})$.
- It is **sunny and hot** outside.

# Proof with a Truth Table: $\neg(x \wedge y) \equiv \neg x \vee \neg y$

$\neg(x \wedge y)$

| $x$ | $y$ | $x \wedge y$ | $\neg(x \wedge y)$ |
|---|---|---|---|
| $T$ | $T$ | $T$ | $F$ |
| $T$ | $F$ | $F$ | $T$ |
| $F$ | $T$ | $F$ | $T$ |
| $F$ | $F$ | $F$ | $T$ |

$\neg x \vee \neg y$

| $x$ | $y$ | $\neg x$ | $\neg y$ | $\neg x \vee \neg y$ |
|---|---|---|---|---|
| $T$ | $T$ | $F$ | $F$ | $F$ |
| $T$ | $F$ | $F$ | $T$ | $T$ |
| $F$ | $T$ | $T$ | $F$ | $T$ |
| $F$ | $F$ | $T$ | $T$ | $T$ |

# Proof with a Truth Table: $\neg(x \vee y) \equiv \neg x \wedge \neg y$

$\neg(x \vee y)$

| $x$ | $y$ | $x \vee y$ | $\neg(x \vee y)$ |
|---|---|---|---|
| $T$ | $T$ | $T$ | $F$ |
| $T$ | $F$ | $T$ | $F$ |
| $F$ | $T$ | $T$ | $F$ |
| $F$ | $F$ | $F$ | $T$ |

$\neg x \wedge \neg y$

| $x$ | $y$ | $\neg x$ | $\neg y$ | $\neg x \wedge \neg y$ |
|---|---|---|---|---|
| $T$ | $T$ | $F$ | $F$ | $F$ |
| $T$ | $F$ | $F$ | $T$ | $F$ |
| $F$ | $T$ | $T$ | $F$ | $F$ |
| $F$ | $F$ | $T$ | $T$ | $T$ |

# Proof with a Truth Table: $\neg(x \wedge y \wedge z) \equiv \neg x \vee \neg y \vee \neg z$

$\neg(x \wedge y \wedge z)$

| $x$ | $y$ | $z$ | $x \wedge y \wedge z$ | $\neg(x \wedge y \wedge z)$ |
|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $F$ |
| $T$ | $T$ | $F$ | $F$ | $T$ |
| $T$ | $F$ | $T$ | $F$ | $T$ |
| $T$ | $F$ | $F$ | $F$ | $T$ |
| $F$ | $T$ | $T$ | $F$ | $T$ |
| $F$ | $T$ | $F$ | $F$ | $T$ |
| $F$ | $F$ | $T$ | $F$ | $T$ |
| $F$ | $F$ | $F$ | $F$ | $T$ |

$\neg x \vee \neg y \vee \neg z$

| $x$ | $y$ | $z$ | $\neg x$ | $\neg y$ | $\neg z$ | $\neg x \vee \neg y \vee \neg z$ |
|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $F$ | $F$ | $F$ | $F$ |
| $T$ | $T$ | $F$ | $F$ | $F$ | $T$ | $T$ |
| $T$ | $F$ | $T$ | $F$ | $T$ | $F$ | $T$ |
| $T$ | $F$ | $F$ | $F$ | $T$ | $T$ | $T$ |
| $F$ | $T$ | $T$ | $T$ | $F$ | $F$ | $T$ |
| $F$ | $T$ | $F$ | $T$ | $F$ | $T$ | $T$ |
| $F$ | $F$ | $T$ | $T$ | $T$ | $F$ | $T$ |
| $F$ | $F$ | $F$ | $T$ | $T$ | $T$ | $T$ |

# Proof with a Truth Table: $\neg(x \vee y \vee z) \equiv \neg x \wedge \neg y \wedge \neg z$

$\neg(x \vee y \vee z)$

| $x$ | $y$ | $z$ | $x \vee y \vee z$ | $\neg(x \vee y \vee z)$ |
|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $F$ |
| $T$ | $T$ | $F$ | $T$ | $F$ |
| $T$ | $F$ | $T$ | $T$ | $F$ |
| $T$ | $F$ | $F$ | $T$ | $F$ |
| $F$ | $T$ | $T$ | $T$ | $F$ |
| $F$ | $T$ | $F$ | $T$ | $F$ |
| $F$ | $F$ | $T$ | $T$ | $F$ |
| $F$ | $F$ | $F$ | $F$ | $T$ |

$\neg x \wedge \neg y \wedge \neg z$

| $x$ | $y$ | $z$ | $\neg x$ | $\neg y$ | $\neg z$ | $\neg x \wedge \neg y \wedge \neg z$ |
|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $F$ | $F$ | $F$ | $F$ |
| $T$ | $T$ | $F$ | $F$ | $F$ | $T$ | $F$ |
| $T$ | $F$ | $T$ | $F$ | $T$ | $F$ | $F$ |
| $T$ | $F$ | $F$ | $F$ | $T$ | $T$ | $F$ |
| $F$ | $T$ | $T$ | $T$ | $F$ | $F$ | $F$ |
| $F$ | $T$ | $F$ | $T$ | $F$ | $T$ | $F$ |
| $F$ | $F$ | $T$ | $T$ | $T$ | $F$ | $F$ |
| $F$ | $F$ | $F$ | $T$ | $T$ | $T$ | $T$ |

# The General De Morgan's Laws

**Complement of Conjunctions $\equiv$ Disjunction of Complements**

$$\left(\bigwedge_{i=1}^{n} x_i\right)' \equiv (x_1 \wedge x_2 \wedge \cdots \wedge x_n)' \equiv (x_1' \vee x_2' \vee \cdots \vee x_n') \equiv \bigvee_{i=1}^{n} x_i'$$

**Complement of Disjunctions $\equiv$ Conjunction of Complements**

$$\left(\bigvee_{i=1}^{n} x_i\right)' \equiv (x_1 \vee x_2 \vee \cdots \vee x_n)' \equiv (x_1' \wedge x_2' \wedge \cdots \wedge x_n') \equiv \bigwedge_{i=1}^{n} x_i'$$

# The General De Morgan's Laws

## Theorem

- Complement of Conjunctions $\equiv$ Disjunction of Complements.

## Proof

- Let $L = (x_1 \wedge x_2 \wedge \cdots \wedge x_n)'$ and $R = (x_1' \vee x_2' \vee \cdots \vee x_n')$
- Prove that $L$ is **TRUE** if and only if $R$ is **TRUE**

$$
\begin{aligned}
L \equiv \textbf{TRUE} \quad & \Leftrightarrow \quad (x_1 \wedge x_2 \wedge \cdots \wedge x_n)' \equiv \textbf{TRUE} \\
& \Leftrightarrow \quad (x_1 \wedge x_2 \wedge \cdots \wedge x_n) \equiv \textbf{FALSE} \\
& \Leftrightarrow \quad (x_i \equiv \textbf{FALSE}) \textbf{ for at least one } i \in \{1, \ldots, n\} \\
& \Leftrightarrow \quad (x_i' \equiv \textbf{TRUE}) \textbf{ for at least one } i \in \{1, \ldots, n\} \\
& \Leftrightarrow \quad (x_1' \vee x_2' \vee \cdots \vee x_n') \equiv \textbf{TRUE} \\
& \Leftrightarrow \quad R \equiv \textbf{TRUE}
\end{aligned}
$$

# The General De Morgan's Laws

**Theorem**

- Complement of Disjunctions $\equiv$ Conjunction of Complements.

**Proof**

- Let $L = (x_1 \vee x_2 \vee \cdots \vee x_n)'$ and $R = (x_1' \wedge x_2' \wedge \cdots \wedge x_n')$
- Prove that $L$ is **TRUE** if and only if $R$ is **TRUE**

$$\begin{aligned}
L \equiv \textbf{TRUE} \quad & \Leftrightarrow \quad (x_1 \vee x_2 \vee \cdots \vee x_n)' \equiv \textbf{TRUE} \\
& \Leftrightarrow \quad (x_1 \vee x_2 \vee \cdots \vee x_n) \equiv \textbf{FALSE} \\
& \Leftrightarrow \quad (x_i \equiv \textbf{FALSE}) \textbf{ for all } i \in \{1, \ldots, n\} \\
& \Leftrightarrow \quad (x_i' \equiv \textbf{TRUE}) \textbf{ for all } i \in \{1, \ldots, n\} \\
& \Leftrightarrow \quad (x_1' \wedge x_2' \wedge \cdots \wedge x_n') \equiv \textbf{TRUE} \\
& \Leftrightarrow \quad R \equiv \textbf{TRUE}
\end{aligned}$$

# Logic puzzles

## The lady or the tiger puzzle

- https://www.youtube.com/watch?v=eUYWXqk5Ags

## Knights and knaves

- https://www.youtube.com/watch?v=C6PeX4iKJbU
- A good introduction to propositional logic

## Fork in the road

- https://www.youtube.com/watch?v=MsV3XRLxX6Q
- https://www.youtube.com/watch?v=F8s03KBjeY0

# The Function $\mathcal{IMPLY}$

**$x$ implies $y$ $\equiv$ if $x$ then $y$**

| $x$ | $y$ | $x \rightarrow y$ | $\neg x \vee y$ |
|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $F$ |
| $F$ | $T$ | $T$ | $T$ |
| $F$ | $F$ | $T$ | $T$ |

**$\mathcal{IMPLY}$ for propositions $P, Q$**

**Conditional:** $P \rightarrow Q$ is equivalent to **Contrapositive:** $\neg Q \rightarrow \neg P$

**Inverse:** $\neg P \rightarrow \neg Q$ is equivalent to **Converse:** $Q \rightarrow P$

# If, Only If, and If and only If (iff)

### The propositions $P$ and $Q$

- $P$: "Passing".
- $Q$: "Good grades".

### The sufficient condition

- $Q \rightarrow P$: "You pass **if** you have good grades".
  - $\mathcal{IMPLY}(T, T) = T$: "You pass with good grades".
  - $\mathcal{IMPLY}(T, F) = F$: "You cannot fail with good grades".
  - $\mathcal{IMPLY}(F, T) = T$: "You might pass with bad grades".
  - $\mathcal{IMPLY}(F, F) = T$: "You might fail with bad grades"

### The equivalent necessary condition

- "You have good grades **only if** you pass".

# If, Only If, and If and only If (iff)

## The propositions $P$ and $Q$

- $P$: "Passing".
- $Q$: "Good grades".

## The necessary condition

- $P \rightarrow Q$: "You pass **only if** you have good grades".
  - $\mathcal{IMPLY}(T, T) = T$: "You might pass with good grades".
  - $\mathcal{IMPLY}(T, F) = F$: "You cannot pass with bad grades".
  - $\mathcal{IMPLY}(F, T) = T$: "You might fail with good grades".
  - $\mathcal{IMPLY}(F, F) = T$: "You fail with bad grades".

## The equivalent sufficient condition

- "You have good grades **if** you pass".

# If, Only If, and If and only If (iff)

**The propositions $P$ and $Q$**

- $P$: “Passing”.
- $Q$: “Good grades”.

**The necessary and sufficient condition**

- $P \equiv Q$: “You pass **if only if** you have good grades”.
  - $\mathcal{EQUIV}(T,T) = T$: “You pass with good grades”.
  - $\mathcal{EQUIV}(T,F) = F$: “You cannot pass with bad grades”.
  - $\mathcal{EQUIV}(F,T) = F$: “You cannot fail with good grades”.
  - $\mathcal{EQUIV}(F,F) = T$: “You fail with bad grades” .

# Be Careful with Your Words!

## Bad Defense Attorney

- Prosecutor: "If the defendants are guilty, then they had accomplices".
- Defense attorney: "This is not true!!"

## What is the defense attorney claiming?

- P: "The defendants are guilty".
- Q: "The defendants had accomplices".
- Prosecutor: $P \rightarrow Q$.
- Defense attorney: $P \rightarrow Q$ is **FALSE**.

## Conclusion

- $P \rightarrow Q$ is **FALSE** only when $P$ is **TRUE** and $Q$ is **FALSE**.
- $P \rightarrow Q$ is **FALSE** only when the defendants are guilty and they did not have accomplices.

# Laws With Two Propositions

## Notations

- Let $P$ and $Q$ be boolean propositions.

## $\mathcal{IMPLY}$

- $P \rightarrow Q \equiv \neg Q \rightarrow \neg P$
- $\neg P \rightarrow \neg Q \equiv Q \rightarrow P$

## $\mathcal{EQUIV}$

- $(P \equiv Q) \equiv (\neg Q \equiv \neg P)$

## $\mathcal{IMPLY}$ and $\mathcal{EQUIV}$

- $(P \rightarrow Q) \wedge (Q \rightarrow P) \equiv (P \equiv Q)$

# $\mathcal{IMPLY}$ and $\mathcal{EQUIV}$

## Theorem

- $(P \rightarrow Q) \wedge (Q \rightarrow P) \equiv (P \equiv Q)$

## Proof with truth tables

| $P$ | $Q$ | $P \rightarrow Q$ | $Q \rightarrow P$ | $(P \rightarrow Q) \wedge (Q \rightarrow P)$ |
|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $T$ | $F$ |
| $F$ | $T$ | $T$ | $F$ | $F$ |
| $F$ | $F$ | $T$ | $T$ | $T$ |

| $P$ | $Q$ | $P \equiv Q$ |
|---|---|---|
| $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ |
| $F$ | $T$ | $F$ |
| $F$ | $F$ | $T$ |

# Laws With Three Propositions

**Notations:**

- Let $P$ and $Q$ and $R$ be three boolean propositions.

**The $\mathcal{IMPLY}$ and $\mathcal{EQUIV}$ transitive laws**

- $((P \rightarrow Q) \wedge (Q \rightarrow R)) \rightarrow (P \rightarrow R)$
- $((P \equiv Q) \wedge (Q \equiv R)) \rightarrow (P \equiv R)$

**The $\mathcal{IMPLY}$ distributive laws**

- $((P \rightarrow Q) \wedge (P \rightarrow R)) \equiv (P \rightarrow (Q \wedge R))$
- $((P \rightarrow Q) \vee (P \rightarrow R)) \equiv (P \rightarrow (Q \vee R))$
- $((Q \rightarrow P) \wedge (R \rightarrow P)) \equiv ((Q \vee R) \rightarrow P)$
- $((Q \rightarrow P) \vee (R \rightarrow P)) \equiv ((Q \wedge R) \rightarrow P)$

# Truth Tables: $((P \to Q) \land (P \to R)) \equiv (P \to (Q \land R))$

## $(P \to Q) \land (P \to R)$

| $P$ | $Q$ | $R$ | $P \to Q$ | $P \to R$ | $(P \to Q) \land (P \to R)$ |
|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $T$ | $F$ | $T$ | $F$ | $F$ |
| $T$ | $F$ | $T$ | $F$ | $T$ | $F$ |
| $T$ | $F$ | $F$ | $F$ | $F$ | $F$ |
| $F$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $F$ | $T$ | $F$ | $T$ | $T$ | $T$ |
| $F$ | $F$ | $T$ | $T$ | $T$ | $T$ |
| $F$ | $F$ | $F$ | $T$ | $T$ | $T$ |

## $P \to (Q \land R)$

| $P$ | $Q$ | $R$ | $Q \land R$ | $P \to (Q \land R)$ |
|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $T$ | $F$ | $F$ | $F$ |
| $T$ | $F$ | $T$ | $F$ | $F$ |
| $T$ | $F$ | $F$ | $F$ | $F$ |
| $F$ | $T$ | $T$ | $T$ | $T$ |
| $F$ | $T$ | $F$ | $F$ | $T$ |
| $F$ | $F$ | $T$ | $F$ | $T$ |
| $F$ | $F$ | $F$ | $F$ | $T$ |

# Truth Tables: $((P \to Q) \vee (P \to R)) \equiv (P \to (Q \vee R))$

## $(P \to Q) \vee (P \to R)$

| $P$ | $Q$ | $R$ | $P \to Q$ | $P \to R$ | $(P \to Q) \vee (P \to R)$ |
|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $T$ | $F$ | $T$ | $F$ | $T$ |
| $T$ | $F$ | $T$ | $F$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $F$ | $F$ | $F$ |
| $F$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $F$ | $T$ | $F$ | $T$ | $T$ | $T$ |
| $F$ | $F$ | $T$ | $T$ | $T$ | $T$ |
| $F$ | $F$ | $F$ | $T$ | $T$ | $T$ |

## $P \to (Q \vee R)$

| $P$ | $Q$ | $R$ | $Q \vee R$ | $P \to (Q \vee R)$ |
|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $T$ | $F$ | $T$ | $T$ |
| $T$ | $F$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $F$ | $F$ |
| $F$ | $T$ | $T$ | $T$ | $T$ |
| $F$ | $T$ | $F$ | $T$ | $T$ |
| $F$ | $F$ | $T$ | $T$ | $T$ |
| $F$ | $F$ | $F$ | $F$ | $T$ |

# Truth Tables: $((Q \to P) \wedge (R \to P)) \equiv ((Q \vee R) \to P)$

## $(Q \to P) \wedge (R \to P)$

| $P$ | $Q$ | $R$ | $Q \to P$ | $R \to P$ | $(Q \to P) \wedge (R \to P)$ |
|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $T$ | $F$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $T$ | $T$ | $T$ |
| $F$ | $T$ | $T$ | $F$ | $F$ | $F$ |
| $F$ | $T$ | $F$ | $F$ | $T$ | $F$ |
| $F$ | $F$ | $T$ | $T$ | $F$ | $F$ |
| $F$ | $F$ | $F$ | $T$ | $T$ | $T$ |

## $(Q \vee R) \to P$

| $P$ | $Q$ | $R$ | $Q \vee R$ | $(Q \vee R) \to P$ |
|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $T$ | $F$ | $T$ | $T$ |
| $T$ | $F$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $F$ | $T$ |
| $F$ | $T$ | $T$ | $T$ | $F$ |
| $F$ | $T$ | $F$ | $T$ | $F$ |
| $F$ | $F$ | $T$ | $T$ | $F$ |
| $F$ | $F$ | $F$ | $F$ | $T$ |

# Truth Tables: $((Q \to P) \vee (R \to P)) \equiv ((Q \wedge R) \to P)$

## $(Q \to P) \vee (R \to P)$

| $P$ | $Q$ | $R$ | $Q \to P$ | $R \to P$ | $(Q \to P) \vee (R \to P)$ |
|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $T$ | $F$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $T$ | $T$ | $T$ |
| $F$ | $T$ | $T$ | $F$ | $F$ | $F$ |
| $F$ | $T$ | $F$ | $F$ | $T$ | $T$ |
| $F$ | $F$ | $T$ | $T$ | $F$ | $T$ |
| $F$ | $F$ | $F$ | $T$ | $T$ | $T$ |

## $(Q \wedge R) \to P$

| $P$ | $Q$ | $R$ | $Q \wedge R$ | $(Q \wedge R) \to P$ |
|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $T$ | $F$ | $F$ | $T$ |
| $T$ | $F$ | $T$ | $F$ | $T$ |
| $T$ | $F$ | $F$ | $F$ | $T$ |
| $F$ | $T$ | $T$ | $T$ | $F$ |
| $F$ | $T$ | $F$ | $F$ | $T$ |
| $F$ | $F$ | $T$ | $F$ | $T$ |
| $F$ | $F$ | $F$ | $F$ | $T$ |

# The $((P \to Q) \wedge (P \to R)) \equiv (P \to (Q \wedge R))$ law

## A "binary" world

- The outside is (**rainy** or **sunny**) and (**cold** or **hot**).
- Alice does not leave the house if it is (**rainy and cold**) outside.

## Notations

- $P$: "staying in the house".
- $Q$: "rainy outside".
- $R$: "cold outside".

## The interpretations of the two equivalent sides of the law

- (If Alice stayed in the house then it is **rainy** outside) **and** (If Alice stayed in the house then it is **cold** outside).
- If Alice stayed in the house then it is (**rainy and cold**) outside.

# The $((P \to Q) \vee (P \to R)) \equiv (P \to (Q \vee R))$ law

## A "binary" world

- The outside is (**rainy** or **sunny**) and (**cold** or **hot**).
- Bob does not leave the house if it is (**rainy or cold**) outside.

## Notations

- $P$: "staying in the house".
- $Q$: "rainy outside".
- $R$: "cold outside".

## The interpretations of the two equivalent sides of the law

- (If Bob stayed in the house then it is **rainy** outside) **or** (If Bob stayed in the house then it is **cold** outside).
- If Bob stayed in the house then it is (**rainy or cold**) outside.

# The $((Q \to P) \wedge (R \to P)) \equiv ((Q \vee R) \to P)$ law

## A "binary" world

- The outside is (**rainy** or **sunny**) and (**cold** or **hot**).
- Alice does not leave the house if it is (**rainy and cold**) outside.

## Notations

- $P$: "leaving the house".
- $Q$: "sunny outside".
- $R$: "hot outside".

## The interpretations of the two equivalent sides of the law

- (If it is **sunny** outside then Alice will leave the house) **and** (If it is **hot** outside then Alice will leave the house).
- If it is (**sunny or hot**) outside then Alice will leave the house.

# The $((Q \to P) \vee (R \to P)) \equiv ((Q \wedge R) \to P)$ law

## A "binary" world

- The outside is (**rainy** or **sunny**) and (**cold** or **hot**).
- Bob does not leave the house if it is (**rainy or cold**) outside.

## Notations

- $P$: "leaving the house".
- $Q$: "sunny outside".
- $R$: "hot outside".

## The interpretations of the two equivalent sides of the law

- (If it is **sunny** outside then Bob will leave the house) **or** (If it is **hot** outside then Bob will leave the house).
- If it is (**sunny and hot**) outside then Bob will leave the house.

# Online Resources

## Why $x \to y$ is TRUE when $x$ is FALSE?

- https://www.youtube.com/watch?v=XjEEEVXqCMo&feature=youtu.be
- https://www.youtube.com/watch?v=xag9TEAORK4&feature=youtu.be

## Necessary and Sufficient Conditions

- https://www.youtube.com/watch?v=QtMFyTV8jfg
- https://www.youtube.com/watch?v=KCMGWRoPuMY
- https://www.youtube.com/watch?v=FHeimgMWiog
- https://www.youtube.com/watch?v=OnbJ8S1Ainc
- https://www.youtube.com/watch?v=fq_DwgmIadw
- https://www.youtube.com/watch?v=ibjL90iY1d0

## The Wason Selection task

- https://www.youtube.com/watch?v=hzS3BOwvT4I

# An Online Tutorial on Conditional Reasoning

- Conditional reasoning and logical equivalence

  https://www.khanacademy.org/test-prep/lsat/lsat-lessons/logic-toolbox-new/a/logic-toolbox--article--conditional-reasoning-logical-equivalence

- If X, then Y — Sufficiency and necessity

  https://www.khanacademy.org/test-prep/lsat/lsat-lessons/logic-toolbox-new/a/logic-toolbox--article--if-x-then-y--sufficiency-and-necessity

- The Logic of “If” vs. “Only if”

  https://www.khanacademy.org/test-prep/lsat/lsat-lessons/logic-toolbox-new/a/logic-toolbox--if-and-only-if

- A quick guide to conditional logic

  https://www.khanacademy.org/test-prep/lsat/lsat-lessons/logic-toolbox-new/a/logic-toolbox--article--quick-guide-conditional-logic

# Complete Sets of Logic Functions

**Definition**

- A set of logic functions is **complete** if any boolean formula can be expressed by using only the functions from that set.

**The canonical complete set**

- The set $\{\mathcal{NOT}, \mathcal{AND}, \mathcal{OR}\}$ ($\{\neg, \wedge, \vee\}$) is complete.
- All $\{1, 2\}$-variable functions can be expressed using $\{\neg, \wedge, \vee\}$.
- Any function with more variables can be expressed by a formula containing one-variable and two-variable functions.

# Expressing Two-Variable Functions With $\{\neg, \vee, \wedge\}$

| Function$(x, y)$ | $(T,T)$ | $(T,F)$ | $(F,T)$ | $(F,F)$ | Formula |
|---|---|---|---|---|---|
| $\mathcal{TRUE}$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $\mathcal{OR}$ | $T$ | $T$ | $T$ | $F$ | $x \vee y$ |
| $\mathcal{Y}$–$\mathcal{IMP}$–$\mathcal{X}$ | $T$ | $T$ | $F$ | $T$ | $x \vee \neg y$ |
| $\mathcal{ID}(\mathcal{X})$ | $T$ | $T$ | $F$ | $F$ | $x$ |
| $\mathcal{X}$–$\mathcal{IMP}$–$\mathcal{Y}$ | $T$ | $F$ | $T$ | $T$ | $\neg x \vee y$ |
| $\mathcal{ID}(\mathcal{Y})$ | $T$ | $F$ | $T$ | $F$ | $y$ |
| $\mathcal{EQUIV}$ | $T$ | $F$ | $F$ | $T$ | $(x \wedge y) \vee (\neg x \wedge \neg y)$ |
| $\mathcal{AND}$ | $T$ | $F$ | $F$ | $F$ | $x \wedge y$ |
| $\mathcal{NAND}$ | $F$ | $T$ | $T$ | $T$ | $\neg(x \wedge y)$ |
| $\mathcal{XOR}$ | $F$ | $T$ | $T$ | $F$ | $(x \vee y) \wedge (\neg x \vee \neg y)$ |
| $\mathcal{NOT}(\mathcal{Y})$ | $F$ | $T$ | $F$ | $T$ | $\neg y$ |
| $\neg(\mathcal{X}$–$\mathcal{IMP}$–$\mathcal{Y})$ | $F$ | $T$ | $F$ | $F$ | $x \wedge \neg y$ |
| $\mathcal{NOT}(\mathcal{X})$ | $F$ | $F$ | $T$ | $T$ | $\neg x$ |
| $\neg(\mathcal{Y}$–$\mathcal{IMP}$–$\mathcal{X})$ | $F$ | $F$ | $T$ | $F$ | $\neg x \wedge y$ |
| $\mathcal{NOR}$ | $F$ | $F$ | $F$ | $T$ | $\neg(x \vee y)$ |
| $\mathcal{FALSE}$ | $F$ | $F$ | $F$ | $F$ | $F$ |

# Other Complete Sets of Logic functions

**How to prove that a set $\mathcal{S}$ of functions is complete?**

- **First method:** Show how the 4 one-variable and 16 two-variable functions can be expressed with functions from $\mathcal{S}$.
- **Second method:** Show that $\mathcal{NOT}$, $\mathcal{AND}$, $\mathcal{OR}$ can be expressed with functions from $\mathcal{S}$ and then transitivity implies that the 4 one-variable and 16 two-variable functions can be expressed with functions from $\mathcal{S}$.

**$\{\mathcal{NOT}, \mathcal{AND}\}$ is a complete set**

- Trivially: $(x \wedge y) \equiv (x \wedge y)$ and $\neg x \equiv \neg x$.
- By one of the De Morgan's laws: $(x \vee y) \equiv \neg(\neg x \wedge \neg y)$.

**$\{\mathcal{NOT}, \mathcal{OR}\}$ is a complete set**

- Trivially: $(x \vee y) \equiv (x \vee y)$ and $\neg x \equiv \neg x$.
- By one of the De Morgan's laws: $(x \wedge y) \equiv \neg(\neg x \vee \neg y)$.

# $\mathcal{NOR}$ ($\downarrow$) is a Complete Function

## The function $\mathcal{NOR}$:

| $x$ | $y$ | $x \downarrow y$ |
|---|---|---|
| $T$ | $T$ | $F$ |
| $T$ | $F$ | $F$ |
| $F$ | $T$ | $F$ |
| $F$ | $F$ | $T$ |

## Expressing $\mathcal{NOT}$, $\mathcal{AND}$, and $\mathcal{OR}$ with $\mathcal{NOR}$:

- $\neg x \equiv x \downarrow x$
- $(x \wedge y) \equiv (x \downarrow x) \downarrow (y \downarrow y)$
- $(x \vee y) \equiv (x \downarrow y) \downarrow (x \downarrow y)$

## Expressing $\mathcal{XOR}$ and $\mathcal{EQUIV}$ with $\mathcal{NOR}$:

- $(x \oplus y) \equiv ((x \downarrow x) \downarrow (y \downarrow y)) \downarrow (x \downarrow y)$
- $(x \equiv y) \equiv (x \downarrow (x \downarrow y)) \downarrow (y \downarrow (x \downarrow y))$

# $\mathcal{NAND}$ ($\uparrow$) is a Complete Function

**The function $\mathcal{NAND}$:**

| $x$ | $y$ | $x \uparrow y$ |
|---|---|---|
| $T$ | $T$ | $F$ |
| $T$ | $F$ | $T$ |
| $F$ | $T$ | $T$ |
| $F$ | $F$ | $T$ |

**Expressing $\mathcal{NOT}$, $\mathcal{AND}$, and $\mathcal{OR}$ with $\mathcal{NAND}$:**

- $\neg x \equiv x \uparrow x$
- $(x \wedge y) \equiv (x \uparrow y) \uparrow (x \uparrow y)$
- $(x \vee y) \equiv (x \uparrow x) \uparrow (y \uparrow y)$

**Expressing $\mathcal{XOR}$ and $\mathcal{EQUIV}$ with $\mathcal{NAND}$:**

- $(x \oplus y) \equiv (x \uparrow (x \uparrow y)) \uparrow (y \uparrow (x \uparrow y))$
- $(x \equiv y) \equiv ((x \uparrow x) \uparrow (y \uparrow y)) \uparrow (x \uparrow y)$

# Quantifiers

## Notations

- Let $P(x)$ be a boolean proposition (**TRUE** or **FALSE**) defined on all the objects $x \in U$.

## The universal quantifier

- **P(x) is TRUE for every $x \in U$** is denoted by:

$$\forall_{x \in U} P(x)$$

## The existential quantifier

- **There exists $x \in U$ such that P(x) is TRUE** is denoted by:

$$\exists_{x \in U} P(x)$$

# Relationships between the quantifiers

## Generalizing the De Morgan's laws

$$\neg(\forall_{x \in U} P(x)) \equiv \exists_{x \in U} \neg P(x)$$

$$\neg(\exists_{x \in U} P(x)) \equiv \forall_{x \in U} \neg P(x)$$

## Example: $P(x)$ means $x$ is smart

- $\neg(\forall_{x \in U} P(x))$: Not everyone is smart.
- $\exists_{x \in U} \neg P(x)$: Someone is not smart.
- $\neg(\exists_{x \in U} P(x))$: There is no one who is smart.
- $\forall_{x \in U} \neg P(x)$: Everyone is not smart.

# Relationships between the quantifiers

**Theorem**

$$\neg(\forall_{x \in U} P(x)) \equiv \exists_{x \in U} \neg P(x)$$

**Proof**

- The left side of the equivalence:
  - $\forall_{x \in U} P(x)$ means that $P(x)$ is **TRUE** for all $x \in U$.
  - $\neg(\forall_{x \in U} P(x))$ means that $P(x)$ is **FALSE** for at least one $x \in U$.
- The right side of the equivalence:
  - $\exists_{x \in U} \neg P(x)$ means that $\neg P(x)$ is **TRUE** for at least one $x \in U$.
  - This is equivalent to $P(x)$ is **FALSE** for at least one $x \in U$.

**Remark**

- This is a generalization of the De Morgan's law:

$$\neg(\bigwedge_{i=1}^{n} P_i) \equiv \bigvee_{i=1}^{n} (\neg P_i)$$

# Relationships between the quantifiers

**Theorem**

$$\neg(\exists_{x \in U} P(x)) \equiv \forall_{x \in U} \neg P(x)$$

**Proof**

- The left side of the equivalence:
  - $\ast$ $\exists_{x \in U} P(x)$ means that $P(x)$ is **TRUE** for at least one $x \in U$.
  - $\ast$ $\neg(\exists_{x \in U} P(x))$ means that $P(x)$ is **FALSE** for all $x \in U$.
- The right side of the equivalence:
  - $\ast$ $\forall_{x \in U} \neg P(x)$ means that $\neg P(x)$ is **TRUE** for all $x \in U$.
  - $\ast$ This is equivalent to $P(x)$ is **FALSE** for all $x \in U$.

**Remark**

- This is a generalization of the De Morgan's law:

$$\neg(\bigvee_{i=1}^{n} P_i) \equiv \bigwedge_{i=1}^{n} (\neg P_i)$$

# Order Among Quantifiers

## Notations

- Let $P(x, y)$ be a boolean proposition (**TRUE** or **FALSE**) defined on all the objects $x, y \in U$.

## Order does not matter

- $\forall_{x \in U} \forall_{y \in U} P(x, y) \equiv \forall_{y \in U} \forall_{x \in U} P(x, y) \equiv \forall_{x,y \in U} P(x, y)$.
- $\exists_{x \in U} \exists_{y \in U} P(x, y) \equiv \exists_{y \in U} \exists_{x \in U} P(x, y) \equiv \exists_{x,y \in U} P(x, y)$.

## Order matters

- $\forall_{x \in U} \exists_{y \in U} P(x, y) \not\equiv \exists_{y \in U} \forall_{x \in U} P(x, y)$.
- A **TRUE** proposition: "For every integer $n$ there exists an integer $m$ such that $m = n^2$" — $\forall_{n \in \mathbb{Z}} \exists_{m \in \mathbb{Z}} (m = n^2)$.
- A **FALSE** proposition: "There exists an integer $m$ such that $m = n^2$ for every integer $n$" — $\exists_{m \in \mathbb{Z}} \forall_{n \in \mathbb{Z}} (m = n^2)$.

# Quantifications of Two Variables

## Summary

| Statement | When true? | When False? |
|---|---|---|
| $\forall_x \forall_y P(x,y)$<br>$\forall_y \forall_x P(x,y)$ | $P(x,y) = T$<br>for every pair $x, y$ | There is a pair $x, y$<br>for which $P(x,y) = F$ |
| $\exists_x \exists_y P(x,y)$<br>$\exists_y \exists_x P(x,y)$ | There is a pair $x, y$<br>for which $P(x,y) = T$ | $P(x,y) = F$<br>for every pair $x, y$ |
| $\forall_x \exists_y P(x,y)$ | For every $x$ there is a $y$<br>for which $P(x,y) = T$ | There is an $x$ such that<br>$P(x,y) = F$ for every $y$ |
| $\exists_x \forall_y P(x,y)$ | There is an $x$ such that<br>$P(x,y) = T$ for every $y$ | For every $x$ there is a $y$<br>for which $P(x,y) = F$ |

# Quantifications of Two Variables

**Example:** $P(x, y)$ **means that** $x$ **and** $y$ **are friends**

- $\forall_{x,y} P(x, y)$:
  - **TRUE**: Everyone is a friend with everyone else.
  - **FALSE**: There exists at least one pair who are not friends.
- $\exists_{x,y} P(x, y)$:
  - **TRUE**: At least one pair are friends.
  - **FALSE**: There are no friends at all.
- $\forall_x \exists_y P(x, y)$:
  - **TRUE**: Everyone has at least one friend.
  - **FALSE**: There exists someone who has no friends.
- $\exists_x \forall_y P(x, y)$:
  - **TRUE**: There exists someone who is a friend with everyone else.
  - **FALSE**: Everyone has at least someone who is not their friend.

# Be Careful and Precise with Logic

## The unexpected hanging paradox

- Paradox with a logical school solution:
  https://www.youtube.com/watch?v=vxlCiV_axQ0
- Paradox with a logical and an epistemological school solutions:
  https://www.youtube.com/watch?v=EPOXhFJsqlM
- Wikipedia:
  https://en.wikipedia.org/wiki/Unexpected_hanging_paradox

# An Online Tutorial

## 8 Logic lectures from TrevTutor

- Propositional Logic: www.youtube.com/watch?v=itrXYg41-V0&feature=youtu.be
- Truth Tables: www.youtube.com/watch?v=UiGu57JzLkE&feature=youtu.be
- Proofs with Truth Tables: www.youtube.com/watch?v=9fX6n0_MDic&feature=youtu.be
- Logic laws: www.youtube.com/watch?v=eihhu72YdpQ&feature=youtu.be
- Conditionals: www.youtube.com/watch?v=xag9TEAORK4&feature=youtu.be
- Proof by Contraposition: https://www.youtube.com/watch?v=X-hJ7krLBn0
- Rules of Inference: www.youtube.com/watch?v=8DW0K3mnc-0&feature=youtu.be
- Predicate Logic: www.youtube.com/watch?v=gyoqX0W-NH4&feature=youtu.be

# Additional Logic Puzzles

## Knights and knaves

- https://www.youtube.com/watch?v=Imgus1ispQk

## Prisoners with hats

- https://youtu.be/N5vJSNXPEwA
- https://www.youtube.com/watch?v=RtidKw-qDxY

## How many liars are at the party?

- https://www.youtube.com/watch?v=jqX9nnRiD9g

## Can you crack the code?

- https://www.youtube.com/watch?v=-etLb-8sHBc&feature=youtu.be