Discrete Structures: Probability

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Discrete Probability

What is probability?

- Probability is a measure of a **likelihood** of an **event** occurring.
  - ∗ The bigger the probability the more likely it is that the event occurs.
  - ∗ The lowest probability is 0 (0%), the highest probability is 1 (100%).
- Probability theory models the phenomenon of chance.
  - ∗ A probabilistic model of random phenomena is defined by assigning **probabilities** to all the possible **outcomes** of an **experiment**.

Learning objectives

- Learn some basic rules, tools, and techniques without formal definitions and proofs.
- Be able to solve probability problems using common sense, intuition, and the basic concepts.

Discrete Probability

A song

- https://www.youtube.com/watch?v=au8hf0-A27s

Likelihood

- https://www.youtube.com/watch?v=QpfMwA0z_1Y

A problem with three methods to solve it

- https://youtu.be/fyv1i2YbNt0

Discrete Probability

Sample spaces

- A **sample space** set $\mathcal{S}$ contains a finite number of **outcomes**

$$\mathcal{S} = \{s_1, s_2, \ldots, s_n\}$$

- Each outcome s_i is associated with a **probability** value $p_i = p(s_i)$ which is a real number between 0 and 1 such that

$$\sum_{i=1}^{n} p_i = 1$$

The uniform probability

- For $1 \leq i \leq n$, $p(s_i) = \frac{1}{n}$ for a probability space $\mathcal{S}$ with n outcomes

$$\sum_{i=1}^{n} \frac{1}{n} = 1$$

Discrete Probability

Intuition and/or motivation and/or explanation

- Suppose the space is **sampled** m times (for a very large m).
- Then s_i is sampled approximately $p_i m$ times for each $1 \leq i \leq n$.

$$\sum_{i=1}^{n} p_i m = \left(\sum_{i=1}^{n} p_i \right) m = m$$

The uniform probability

- Suppose the space is **sampled** m times (for a very large m).
- Then s_i is sampled approximately $\frac{m}{n}$ times for each $1 \leq i \leq n$.

$$\sum_{i=1}^{n} \frac{m}{n} = \left(\sum_{i=1}^{n} \frac{1}{n} \right) m = m$$

Discrete Probability

Generating sample spaces

- An **experiment** is defined that ends up with one outcome taken from the sample space.
- Each outcome is associated with a probability value.

Four canonical examples

- **Flipping** (or **tossing**) a coin.
- **Throwing** (or **rolling**) a dice.
- **Drawing** a card from a deck of cards.
- **Drawing** a marble from a bag of marbles.

Flipping One Coin

Model

- **Experiment:** Flip a coin that has two sides, one is **Heads** (**H**) and the other is **Tails** (**T**), to observe one of the sides.
- **Sample space:** $\{\mathbf{H}, \mathbf{T}\}$.

Fair and biased coins

- In a **fair coin** the probabilities of the two outcomes are the same:

$$p(\mathbf{H}) = p(\mathbf{T}) = 1/2$$

- For a real number $0 \leq p \leq 1$, in a **biased coin** the probability of flipping **H** is p while the probability of flipping **T** is $1 - p$:

$$p(\mathbf{H}) + p(\mathbf{T}) = p + (1 - p) = 1$$

Online coins

- `https://www.geogebra.org/m/LZbwMZtJ`

Flipping Several Coins or One Coin Several Times

Flipping two coins together or one coin twice

- The sample space has **4** outcomes:

$$\{(\mathbf{H}, \mathbf{H}), (\mathbf{H}, \mathbf{T}), (\mathbf{T}, \mathbf{H}), (\mathbf{T}, \mathbf{T})\}$$

- With fair coins the probability of each outcome is $\mathbf{1/4}$.

Flipping three coins together or one coin three times

- The sample space has **8** outcomes:

$$\{(\mathbf{H}, \mathbf{H}, \mathbf{H}), (\mathbf{H}, \mathbf{H}, \mathbf{T}), (\mathbf{H}, \mathbf{T}, \mathbf{H}), (\mathbf{H}, \mathbf{T}, \mathbf{T}), (\mathbf{T}, \mathbf{H}, \mathbf{H}), (\mathbf{T}, \mathbf{H}, \mathbf{T}), (\mathbf{T}, \mathbf{T}, \mathbf{H}), (\mathbf{T}, \mathbf{T}, \mathbf{T})\}$$

- With fair coins the probability of each outcome is $\mathbf{1/8}$.

Flipping *k* coins together or one coin *k* times

- The sample space has $\mathbf{2^k}$ outcomes:

$$\{(\mathbf{H}, \mathbf{H} \ldots, \mathbf{H}, \mathbf{H}), (\mathbf{H}, \mathbf{H}, \ldots, \mathbf{H}, \mathbf{T}), \ldots, (\mathbf{T}, \mathbf{T}, \ldots, \mathbf{T}, \mathbf{H}), (\mathbf{T}, \mathbf{T}, \ldots, \mathbf{T}, \mathbf{T})\}$$

- With fair coins the probability of each outcome is $\mathbf{(1/2)^k}$.

Throwing One Dice

Model

- **Experiment:** Throw a dice that has 6 faces (sides) labeled by the numbers $1, 2, 3, 4, 5, 6$ to observe one of the faces.
- **Sample space:** $\{1, 2, 3, 4, 5, 6\}$.
- In a fair dice the probability of each outcome is $1/6$.

A generalized dice

- For an integer $n \geq 2$, the generalized dice has n faces labeled by the numbers $1, 2, \ldots, n$.
- The **sample space** is $\{1, 2, \ldots, n\}$.
- In a fair generalized dice the probability of each outcome is $1/n$.
- A coin is a generalized dice with $n = 2$ faces.

Throwing Several Dice or One Dice Several Times

Throwing two dice together or one dice twice

- There are $\mathbf{6^2} = \mathbf{36}$ outcomes:

$$\{(1,1),(1,2),\ldots,(3,6),(4,1),\ldots,(6,5),(6,6)\}$$

Throwing k dice together or one dice k times

- There are $\mathbf{6^k}$ outcomes:

$$\{(1,1,\ldots,1),(1,1,\ldots,2),\ldots,(2,1,\ldots,1),\ldots,(6,6,\ldots,5),(6,6,\ldots,6)\}$$

Online generalized dice

- `https://rolladie.net/#!numbers=2&high=6&length=1&sets=&addfilters=&last_roll_only=false&totals_only=false`

Drawing Cards

Model

- **Sample space:**
  - A deck of cards contains 52 cards.
  - There are 4 suits: 13 Black Clubs ($\clubsuit$), 13 Red Diamonds ($\diamondsuit$), 13 Red Hearts ($\heartsuit$), and 13 Black Spades ($\spadesuit$).
  - Each suit has nine number cards $2, 3, 4, 5, 6, 7, 8, 9, 10$, three face cards Jack (J), Queen (Q), and King (K), and one Ace (A).
- **Experiment:** Blindly draw a random card from the deck .
- The probability of drawing any of the cards is $1/52$.

Drawing more than one card from the deck

- When k cards are drawn together the probability of each combination is $1/\binom{52}{k}$.

Online cards

- https://www.calculatorsoup.com/calculators/statistics/random-card-generator.php

Drawing Colored Pebbles from a Bag

Model

- **Sample space:**
  - ∗ A bag contains n pebbles each is colored with one color out of k possible colors.
  - ∗ For $1 \leq i \leq k$, there are n_i pebbles of color i.
  - ∗ $n = n_1 + n_2 + \cdots + n_k$.
- **Experiment:** Blindly draw a random pebble from the bag.
- The probability of each pebble to be drawn is $1/n$.

Drawing more than one pebble from the bag

- **Model I:** The pebble is returned to the bag before drawing the next pebble.
- **Model II:** The pebble is left outside before drawing the next pebble.

Events

Definition

- An **event** $\mathcal{E}$ is a set of outcomes from the probability space:

$$\mathcal{E} \subseteq \mathcal{S}$$

- If an event $\mathcal{E}$ contains the k outcomes $s_{i_1}, s_{i_2}, \ldots, s_{i_k}$ then its probability is

$$P(\mathcal{E}) = \sum_{j=1}^{n} p(s_{i_j})$$

- For a sample space with n outcomes, when all the outcomes have the same probability then

$$P(\mathcal{E}) = \frac{|\mathcal{E}|}{|\mathcal{S}|} = \frac{k}{n}$$

Events

The null event

- The **null** event $\mathcal{E} = \emptyset$ contains no outcomes.
- The probability of the null event is $p(\emptyset) = 0$.

The all event

- The **all** event $\mathcal{E} = \mathcal{S}$ contains all possible outcomes.
- The probability of the all event is $p(\mathcal{S}) = 1$.

The singleton events

- A **singleton** event $\mathcal{E} = \{s_i\}$ contains only one outcome.
- The probability of the singleton event $\{s_i\}$ is $p(\{s_i\}) = p_i$.
- When all the n outcomes of the sample space have the same probability then the probability of the singleton event $\{s_i\}$ is $1/n$.

Examples of Events

Coins

- Let A be the event of observing at least one Heads when flipping two fair coins.
- $A = \{\mathbf{HH}, \mathbf{HT}, \mathbf{TH}\}$.
- $p(A) = 3/4$ because A contains 3 outcomes out of the 4 possible outcomes.

Dice

- Let B be the event of observing an even number when throwing a 6-face fair dice.
- $B = \{2, 4, 6\}$.
- $p(B) = 3/6 = 1/2$ because B contains 3 outcomes out of the 6 possible outcomes.

Examples of Events

Cards

- Let C be the event of drawing an Ace from a standard 52-card deck.
- $C = \{A\clubsuit, A\diamondsuit, A\heartsuit, A\spadesuit\}$.
- $p(C) = 4/52 = 1/13$ because C contains 4 outcomes out of the 52 possible outcomes.

Pebbles

- Let D be the event of drawing a **Red** pebble from a bag containing 3 **Red** pebbles, 4 **Blue** pebbles, and 5 **Green** pebbles.
- $D = \{1\mathbf{R}, 2\mathbf{R}, 3\mathbf{R}\}$.
- $p(D) = 3/12 = 1/4$ because D contains 3 outcomes out of the $12 = 3 + 4 + 5$ possible outcomes.

The Complement of an Event

Definition

- Let $\mathcal{E}$ be an event.
- Event $\overline{\mathcal{E}}$ occurs if and only if event $\mathcal{E}$ does not occur.

Observation

- $p(\overline{\mathcal{E}}) = 1 - p(\mathcal{E})$.
- Proof idea: Any outcome either belongs to $\mathcal{E}$ or belongs to $\overline{\mathcal{E}}$.

Example

- A: The event that a dice shows an odd number.
- $\overline{A}$: The event that a dice shows an even number.
- $p(A) = p(\overline{A}) = 1/2$.

The Union of Two Events

Definition

- The union event $A \cup B$ occurs if and only if event A occurs **or** event B occurs (or both).

Disjoint events

- Events A and B are **disjoint** (**mutually exclusive**) if $A \cap B = \emptyset$.

Probability of disjoint events A and B

$$p(A \cup B) = p(A) + p(B)$$

The Union of Two Events

Example: throw a 6-face fair dice

- Let $A = \{2, 4, 6\}$ be the event of observing an even number, its probability is $p(A) = 3/6 = 1/2$.
- Let $B = \{3, 5\}$ be the event of observing an odd prime number, its probability is $p(B) = 2/6 = 1/3$.
- Let $A \cup B = \{2, 3, 4, 5, 6\}$ be the event of observing either an even number or an odd prime number, its probability is $p(A \cup B) = 5/6$.
- A and B are disjoint events because $A \cap B = \emptyset$ and therefore

$$p(A \cup B) = p(A) + p(B) = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$$

The Intersection of Events

Definition

- The intersection event $A \cap B$ occurs if and only if both event A **and** event B occur.

Dependent and Independent events

- Informally, events A and B are **dependent** if the occurrence of one of them "influences" the probability of the other one occurring.
- Events A and B are **independent** if the probability of event B occurring in the sample space A is the same as its probability of occurring in the overall sample space.

Probability of independent events A and B

$$p(B) = \frac{p(A \cap B)}{p(A)} \implies p(A \cap B) = p(A) \cdot p(B)$$

The Intersection of Events

Example: throw two 6-face fair dice

- Let $A = \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6)\}$ be the event of observing 1 on the first dice, its probability is $p(A) = 6/36 = 1/6$.
- Let $B = \{(1,6),(2,6),(3,6),(4,6),(5,6),(6,6)\}$ be the event of observing 6 on the second dice, its probability is $p(B) = 6/36 = 1/6$.
- The intersection event $A \cap B = \{(1,6)\}$ is the event of observing both 1 on the first dice and 6 on the second dice, its probability is $p(A \cap B) = 1/36$.
- Intuitively both events are independent because they cannot "influence" each other.
- Indeed,

$$p(A \cap B) = p(A) \cdot p(B) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$$

The Intersection of Events

Example: throw a 6-face fair dice

- Let $A = \{1, 2, 3\}$ be the event of observing a smaller than 4 number, its probability is $p(A) = 3/6 = 1/2$.
- Let $B = \{1, 2, 3, 4\}$ be the event of observing a smaller than 5 number, its probability is $p(B) = 4/6 = 2/3$.
- Let $C = \{2, 3, 5\}$ be the event of observing a prime number, its probability is $p(C) = 3/6 = 1/2$.
- A and C are dependent events because $p(A \cap C) = p(\{2, 3\}) = 2/6 = 1/3$ while $p(A) \cdot p(C) = (1/2) \cdot (1/2) = 1/4$.
- B and C are independent events because $p(B \cap C) = p(\{2, 3\}) = 2/6 = 1/3$ and also $p(B) \cdot p(C) = (2/3) \cdot (1/2) = 1/3$.

The Principle of Inclusion exclusion

Theorem

- $p(A \cup B) = p(A) + p(B) - p(A \cap B)$ for any two events A and B.

Example: throw a 6-face fair dice

- Let $A = \{1, 3, 5\}$ be the event of observing an odd number, its probability is $p(A) = 3/6 = 1/2$.
- Let $B = \{2, 3, 5\}$ be the event of observing a prime number, its probability is $p(B) = 3/6 = 1/2$.
- Let $A \cup B = \{1, 2, 3, 5\}$ be the event of observing either an odd number or a prime number, its probability is $p(A \cup B) = 4/6 = 2/3$.
- Let $A \cap B = \{3, 5\}$ be the event of observing an odd prime number, its probability is $p(A \cap B) = 2/6 = 1/3$.
- By the principle of inclusion exclusion

$$p(A \cup B) = p(A) + p(B) - p(A \cap B) = \frac{1}{2} + \frac{1}{2} - \frac{1}{3} = \frac{2}{3}$$

Remarks

Many events and laws among sets

- Intersection and union of more than two events are generalized in a “natural” way.
- Events are sets and therefore the laws of sets apply to their probabilities. For example, the associative laws, the distributive laws, and De Morgan’s laws.

Two main techniques for calculating probabilities of events

- When all the outcomes have the same probability, the **counting** method counts the number of outcomes that belong to the event and divides it by the sample space size.
- The **probability** method applies the probability rules like the complement, union, and intersection among events.

Some Online Examples

Probability and counting: guessing in T/F exams

- https://www.youtube.com/watch?v=3V2omKRX9gc

A dice problem

- https://www.youtube.com/watch?v=Kgudt4PXs28&feature=youtu.be

The Law of Large Numbers

Setting

- Let $\mathcal{S} = \{s_1, s_2, \ldots, s_n\}$ be a sample space in which the probability of each outcome s_i is p_i for all $1 \leq i \leq n$.
- Suppose the space $\mathcal{S}$ is **sampled** m times (for a very large m).

Informal statement of the law

- s_i is sampled **approximately** $p_i m$ times for each $1 \leq i \leq n$.
- With the uniform probability, s_i is sampled **approximately** $\frac{m}{n}$ times for each $1 \leq i \leq n$.

The other direction

- For $1 \leq i \leq n$, suppose s_i is sampled h_i times.
- Then $p_i = \frac{h_i}{m}$.

The Law of Large Numbers and Simulations

Introducing the concept

- https://www.youtube.com/watch?v=MntX3zWNWec

Experimental versus theoretical probability

- https://www.youtube.com/watch?v=Nos-xOCpQqg&feature=youtu.be

Collecting six coupons via simulations

- https://www.youtube.com/watch?v=2AVLfSRpmfg&feature=youtu.be

Conditional Probability

Motivation

- The probabilities assiciated with the outcomes are **pure** in the sense that they predict their likelihood assuming nothing is known about the saple space.
- These predicting probabilities may change once some information is known about some events that had happened.

Definition

- **Conditional probability** is defined as the likelihood of an event occurring based on the occurrence of a previous event.

Notation

- Let A and B be two events in the sample space $\mathcal{S}$. The event $B|A$ in $\mathcal{S}$ is the event B **given** that event A happened.

Conditional Probability

Formula for two events A and B

$$p(B|A) = \frac{p(A \cap B)}{p(A)} \qquad p(A|B) = \frac{p(A \cap B)}{p(B)}$$

Bayes' Theorem for two events A and B

$$p(B|A) \cdot P(A) = p(A|B) \cdot P(B)$$

Two independent events A and B

- If A and B are independent events then $p(A \cap B) = p(A) \cdot p(B)$.
- Therefore

$$p(B|A) = \frac{p(A \cap B)}{p(A)} = \frac{p(A) \cdot p(B)}{p(A)} = p(B)$$

$$p(A|B) = \frac{p(A \cap B)}{p(B)} = \frac{p(A) \cdot p(B)}{p(B)} = p(A)$$

Conditional Probability

Example: throw a 6-face fair dice

- Let $A = \{1, 2, 3, 4, 5\}$ be the event of observing a number smaller than 6, its probability is $p(A) = 5/6$.
- Let $B = \{1, 3, 5\}$ be the event of observing an odd number, its probability is $p(B) = 3/6 = 1/2$.
- The intersection event $A \cap B = \{1, 3, 5\}$ is observing an odd number smaller than 6, its probability is $p(A \cap B) = 3/6 = 1/2$.
- Let $B|A$ be the event of observing an odd number **given** that event A happened:
$$p(B|A) = \frac{p(A \cap B)}{p(A)} = \frac{1/2}{5/6} = \frac{3}{5}$$
- Let $A|B$ be the event of observing a number smaller than 6 **given** that event B happened:
$$p(A|B) = \frac{p(A \cap B)}{p(B)} = \frac{1/2}{1/2} = 1$$

Conditional Probability

Example: flip two fair coins (sample space is $\{HH, HT, TH, TT\}$)

- Let $A = \{HH, HT, TH\}$ be the event of flipping at least one heads, its probability is $p(A) = 3/4$.
- Let $B = \{TH, TT\}$ be the event of the first coin showing tails, its probability is $p(B) = 2/4 = 1/2$.
- The intersection event $A \cap B = \{TH\}$ is flipping at least one heads while the first coing showing tails, its probability is $p(A \cap B) = 1/4$.
- Let $B|A$ be the event of the first coin showing tails **given** that event A happened:
$$p(B|A) = \frac{p(A \cap B)}{p(A)} = \frac{1/4}{3/4} = \frac{1}{3}$$
- Let $A|B$ be the event of flipping at least one heads **given** that event B happened:
$$p(A|B) = \frac{p(A \cap B)}{p(B)} = \frac{1/4}{1/2} = \frac{1}{2}$$

Conditional Probability: Online Examples

Statistics example: independence and conditional probability

- https://youtu.be/pIfpHdGVwLU

Chance of guessing correctly

- https://www.youtube.com/watch?v=7DSA4qQv3oc

Coin example: conditional probability

- https://youtu.be/Zxm4Xxvzohk

The false positive “paradox”

- https://youtu.be/1csFTDXXULY

Conditional probability via false positive and false negative

- https://youtu.be/hxEdXUB_IdQ

Intuition Or Not?

The Boy and a Girl Paradox

- https://www.youtube.com/watch?v=YtK4R66_YAk
- https://www.youtube.com/watch?v=GYpvh5NBYuI
- https://www.youtube.com/watch?v=O9YkN2UdfFE

Variations

- Three red-blue cards: https://www.youtube.com/watch?v=1IFM1ngwEb0
- 2 Aces and 2 twos: https://www.youtube.com/watch?v=TGukPgViEkg
- A dice version: https://www.youtube.com/watch?v=lTJydnfGyyk

The Monty Hall Problem

- https://youtu.be/_X5erR9LKUs
- https://youtu.be/4Lb-6rxZxx0

Problems With "Counter Intuitive" Answers

Relatively Easier Problems

- Random walkers meeting on a grid
  - https://www.youtube.com/watch?v=F_kt51Qj1RI
- Guessing randomly on A matching test
  - https://www.youtube.com/watch?v=h2VMMlpEtRU

Relatively Harder Problems

- What is the chance twin brothers are identical?
  - https://www.youtube.com/watch?v=xBPMoavS5Hs
- HH needs 6 and HT needs 4 flips via markov chains
  - https://www.youtube.com/watch?v=IAiNqQi30-Y
  - https://www.youtube.com/watch?v=--mxW3jDlGk
- Conditional probability with boys and girls
  - https://www.youtube.com/watch?v=UQnBGm1aLjY

Beyond Basic Probability

Fermet needs 2 wins Euler needs 3 wins

- https://www.youtube.com/watch?v=C_nV3cVNjog

Weird dice with the same distribution

- https://www.youtube.com/watch?v=kG4Fh_PrN-E

The reason casinos always win meet the law of large numbers

- https://www.youtube.com/watch?v=RXY-WN0ahiw

An alien extinction riddle

- https://www.youtube.com/watch?v=A5-Q2GdD5xw