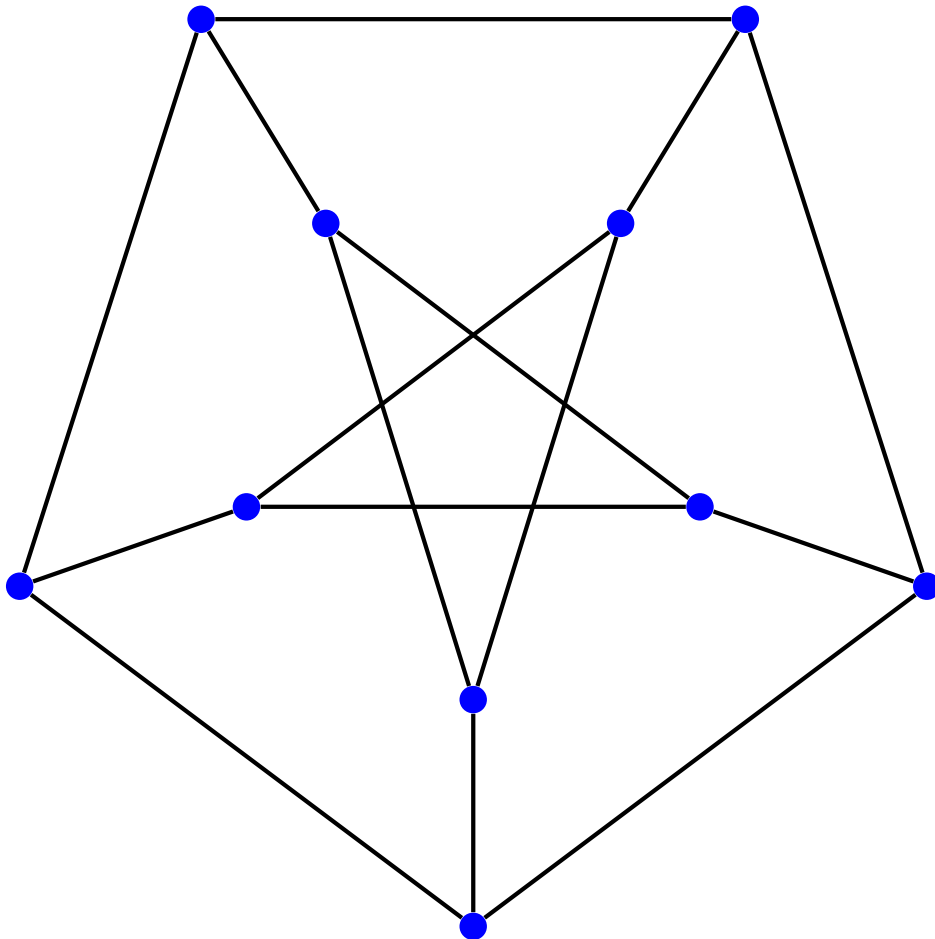


Discrete Structures

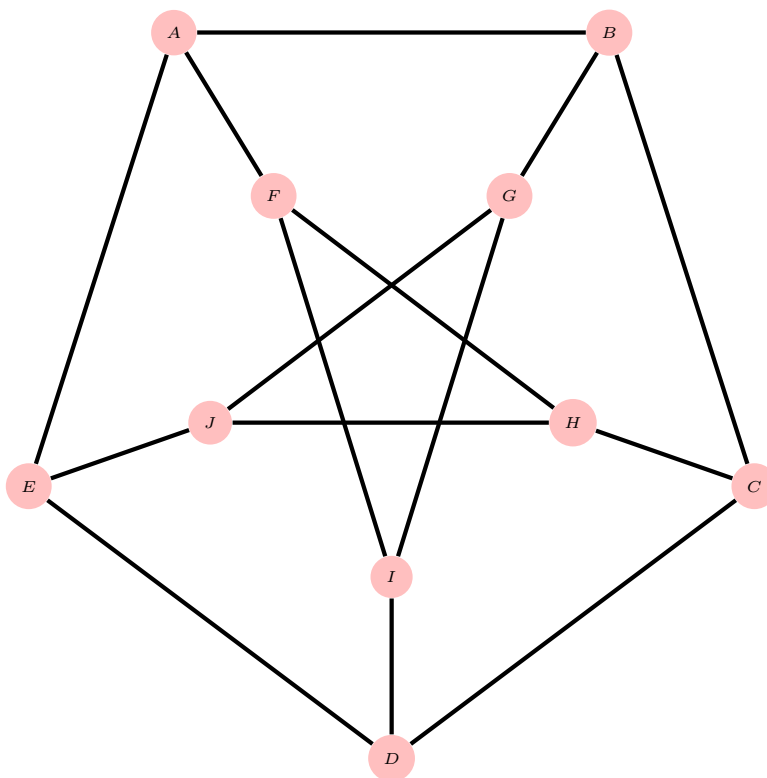
Graphs Practice Problems

Name and ID:

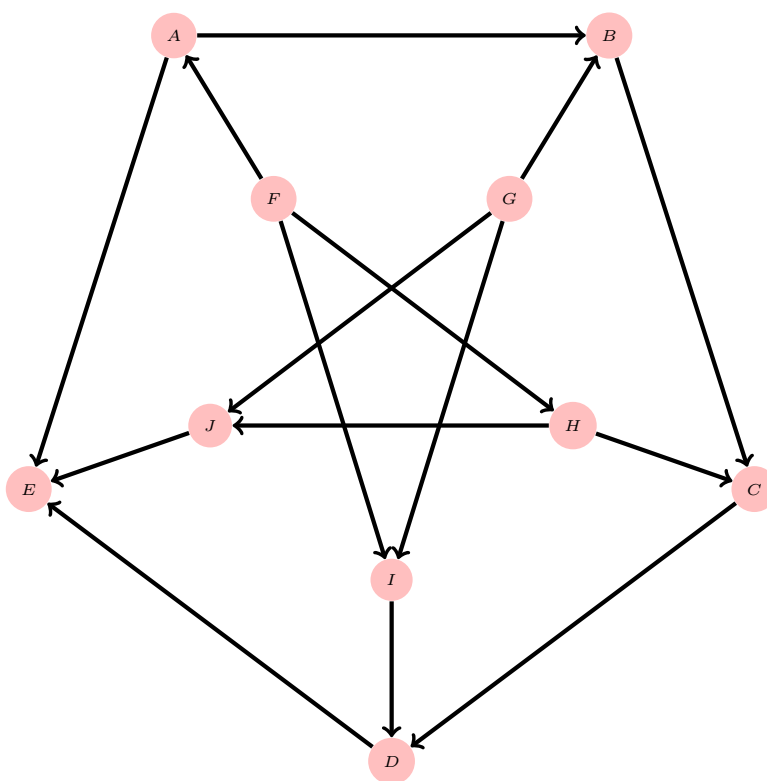


1. Find two additional “nice” drawings for the Petersen graph.

2. (a) Construct the adjacency matrix of the following undirected Petersen graph.
 (b) Construct the adjacency lists of the following undirected Petersen graph.



3. (a) Construct the adjacency matrix of the following directed Petersen graph.
 (b) Construct the incoming and outgoing adjacency lists of the following directed Petersen graph.

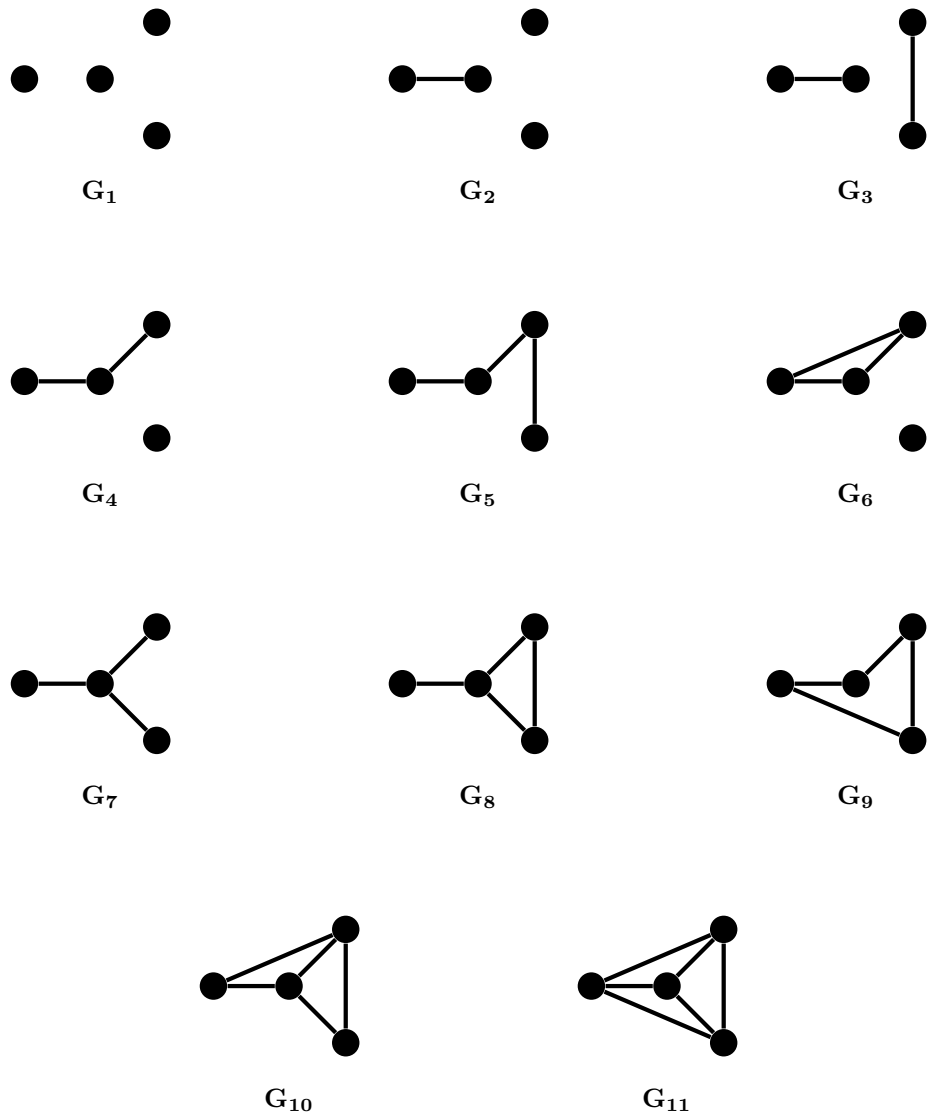


4. A simple undirected **labeled** graph H is a **labeled-sub-graph** of another simple undirected **labeled** graph G if both G and H have the same set of vertices and every edge of H is also an edge of G (however there could be edges of G that are not edges of H).
- For $n \geq 1$, let G and H be two simple undirected labeled graphs on the vertices $\{1, 2, \dots, n\}$. Assume that both graphs are represented by adjacency matrices.
- Describe how to determine if H is a labeled-sub-graph of G .
- Justify the correctness of your solution.
5. The **union** graph of two simple undirected labeled graphs with the same set of vertices is a simple undirected labeled graph with the same vertex set that contains all of the edges that belong to at least one of the graphs.
- For $n \geq 1$, let G and H be two simple undirected labeled graphs on the vertices $\{1, 2, \dots, n\}$. Assume that these graphs are represented by adjacency matrices.
- Describe how to construct the adjacency matrix of the union graph of G and H .
- Justify the correctness of your construction.
6. The **intersection** graph of two simple undirected labeled graphs with the same set of vertices is a simple undirected labeled graph with the same vertex set that contains all of the edges that belong to both graphs.
- For $n \geq 1$, let G and H be two simple undirected labeled graphs on the vertices $\{1, 2, \dots, n\}$. Assume that these graphs are represented by adjacency matrices.
- Describe how to construct the adjacency matrix of the intersection graph of G and H .
- Justify the correctness of your construction.
7. Two simple (no self loops and no parallel edges) undirected graphs G and H are **isomorphic** if there exists a one-to-one function f from the vertices of G to the vertices of H such that an edge (u, v) exists in G **if and only if** the edge $(f(u), f(v))$ exists in H .
- For each one of the following four parts determine if G and H are (i) always isomorphic, or (ii) could never be isomorphic, or (iii) sometimes isomorphic and sometimes not isomorphic.
- Justify your answers.
- G and H have the same number of edges. However, G has n vertices while H has $n + 1$ vertices for $n \geq 1$.
 - Both graphs have $n \geq 2$ vertices and exactly one edge.
 - Both graphs have $n \geq 4$ vertices and exactly two edges.
 - Both graphs have $n \geq 2$ vertices and all possible edges exist.
8. Let G and H be two simple undirected graphs, each with 6 vertices and the **same** number of edges.
- In each of the following parts, if it is possible to find two non-isomorphic graphs G and H that satisfy the conditions, draw examples for G and H and explain why G and H are not isomorphic. If it is not possible, explain why it is not possible to find two such graphs.
- G and H are not isomorphic.
 - G and H are not isomorphic and every vertex has degree 5.
 - G and H are not isomorphic and every vertex has degree 2.
 - G and H are not isomorphic and every vertex has degree 1.
9. In the following questions explain why the two graphs you drew are not isomorphic.
- Draw two non-isomorphic simple graphs with 4 vertices and 2 edges.
 - Draw two non-isomorphic simple bipartite graphs with 4 vertices and 3 edges.
 - Draw two non-isomorphic simple graphs with 6 vertices and 6 edges for which the degrees of all the vertices are 2.
 - Draw two non-isomorphic simple trees with 6 vertices and 3 leaves (vertices whose degree is 1).

10. Below are all the 11 non-isomorphic graphs with 4 vertices.

- (a) Identify all the graphs that are connected.
- (b) Identify all the graphs that are forests.
- (c) Identify all the graphs that are trees.
- (d) Identify all the graphs that are regular (all vertices have the same degree).
- (e) Identify all the graphs that are bipartite.

Fill the table below with Yes (Y) and No (N).

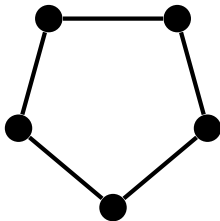


	G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_8	G_9	G_{10}	G_{11}
connected graphs											
forests											
trees											
regular graphs											
bipartite											

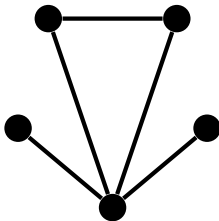
11. Below are all the 6 non-isomorphic graphs with 5 vertices and 5 edges.

- (a) Identify all the graphs that are connected.
- (b) Identify all the graphs that are regular (all vertices have the same degree).
- (c) Identify all the graphs that are bipartite.

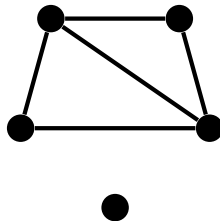
Fill the table below with Yes (Y) and No (N).



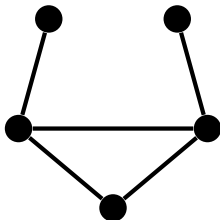
H_1



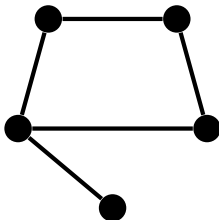
H_2



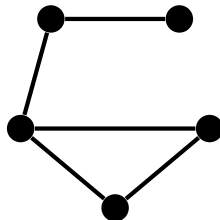
H_3



H_4



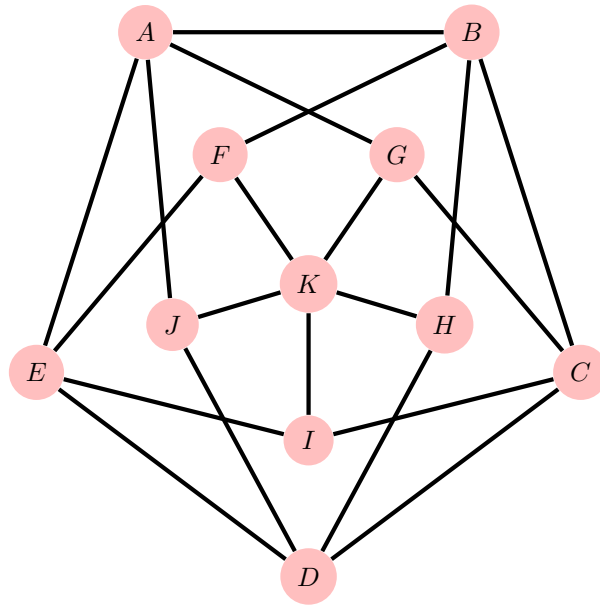
H_5



H_6

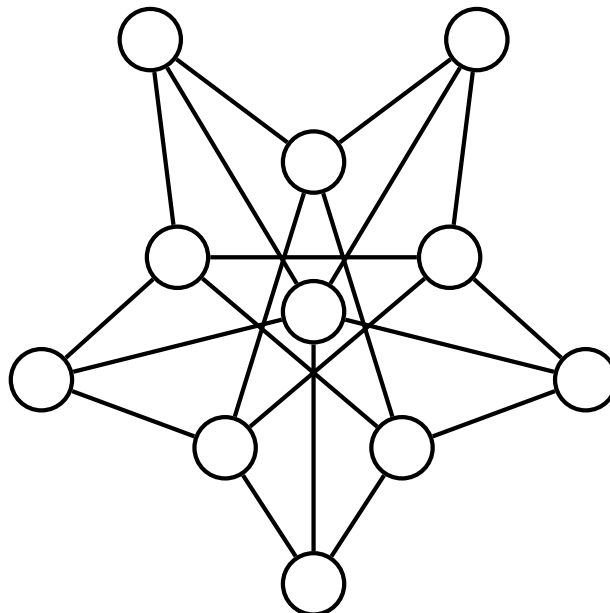
	H_1	H_2	H_3	H_4	H_5	H_6
connected graphs						
regular graphs						
bipartite						

12. Consider the following Grötzsch graph with the 11 vertices $A, B, C, D, E, F, G, H, I, J, K$.

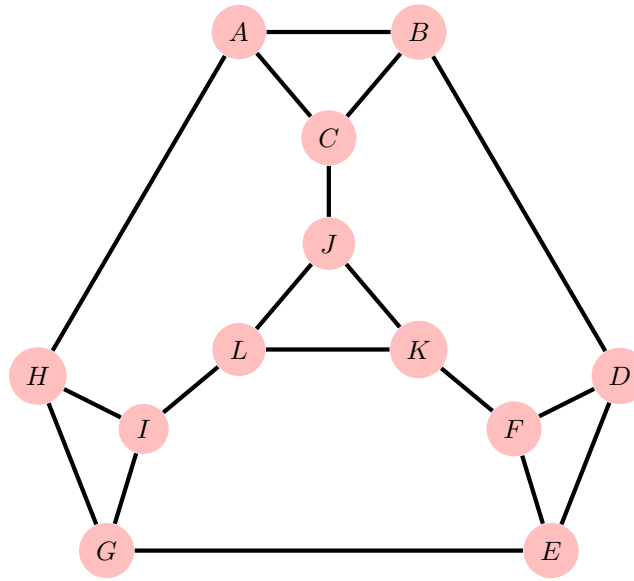


Justify your answers to the following questions and task.

- Construct the adjacency matrix of the Grötzsch graph.
- Construct the adjacency lists of the Grötzsch graph.
- Is the Grötzsch graph connected?
- Is the Grötzsch graph a tree?
- Is the Grötzsch graph a bipartite graph?
- Assign the 11 labels A, B, \dots, K to the 11 vertices in the drawing below to prove that this drawing is isomorphic to the above drawing of the Grötzsch graph. That is, for any pair of vertices $u, v \in \{A, B, \dots, K\}$, the edge (u, v) exists in the original drawing if and only if it exists in the drawing below with your label-assignment.

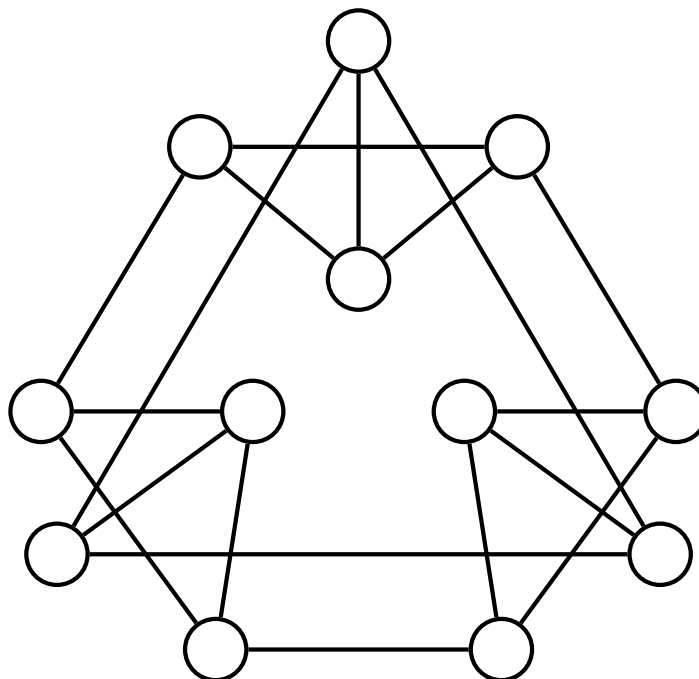


13. Consider the following FourTriangles graph with the 12 vertices $A, B, C, D, E, F, G, H, I, J, K, L$.



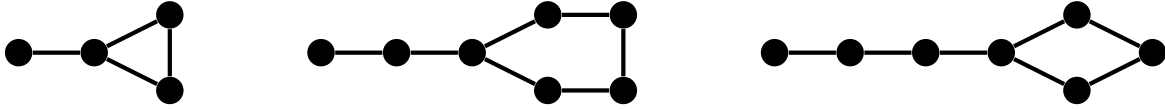
Justify your answers to the following questions and task.

- Construct the adjacency matrix of the FourTriangles graph.
- Construct the adjacency lists of the FourTriangles graph.
- Is the FourTriangles graph connected?
- Is the FourTriangles graph a tree?
- Is the FourTriangles graph a bipartite graph?
- Assign the 12 labels A, B, \dots, L to the 12 vertices in the drawing below to prove that this drawing is isomorphic to the above drawing of the FourTriangles graph. That is, for any pair of vertices $u, v \in \{A, B, \dots, L\}$, the edge (u, v) exists in the original drawing if and only if it exists in the drawing below with your label-assignment.



14. A **balloon-graph** is a graph that is composed of a cycle that is connected to a path.

- All the edges of the graph are either part of the cycle or part of the path.
- Exactly one vertex is shared by the cycle and the path.
- The cycle has at least 3 vertices and therefore at least 2 vertices belong only to the cycle.
- The path has at least 2 vertices and therefore at least 1 vertex belong only to the path.

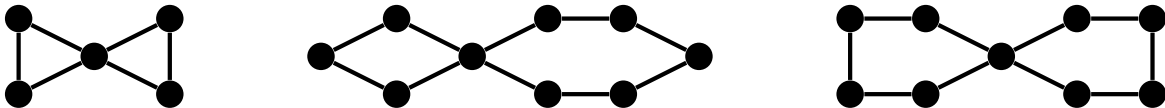


A balloon-graph must contain at least 4 vertices. See above the only possible balloon-graph with 4 vertices and two of the four possible non-isomorphic balloon-graphs with 7 vertices. One of them has a path of length 3 that is connected to a cycle of size 5 and one of them has a path of length 4 that is connected to a cycle of size 4.

- Illustrate the possible non-isomorphic balloon-graphs with 4, 5, 6, and 7 vertices.
- For $n \geq 4$, how many non-isomorphic balloon-graphs with n vertices exist? Justify your answer.
- For $n \geq 4$, all balloon-graphs with n vertices have the same number of edges. What is this number? Prove your answer.
- For $n \geq 4$, the degree sequence of all possible balloon-graphs is the same. Let G be a balloon-graph with n vertices. How many vertices in G have degree 0? How many vertices in G have degree 1? How many vertices in G have degree 2? How many vertices in G have degree 3? How many vertices in G have degree greater than 3? Justify your answers.

15. An **eight-graph** is a graph that is composed of two cycles that share exactly 1 vertex.

- All the edges of the graph are part of one of the two cycles.
- Exactly one vertex is shared by both cycles.
- The rest of the vertices belong only to one of the cycles.
- A cycle must contain at least 3 vertices including the one that connects both cycles.



An eight-graph must contain at least 5 vertices. See above the only possible eight-graph with 5 vertices and two of the three possible non-isomorphic eight-graphs with 9 vertices. One of them has a cycle of size 4 that is connected to a cycle of size 6 and one of them has two connected cycles of size 5.

- Illustrate the only possible non-isomorphic eight-graphs with 5, 6, 7, 8, and 9 vertices.
- For $n \geq 5$, how many non-isomorphic eight-graphs with n vertices exist? Justify your answer.
Hint: The number is the same for the odd number $2k - 1$ and the even number $2k$. Find this number as a function of k which is $\lceil n/2 \rceil$.
- For $n \geq 5$, all eight-graphs with n vertices have the same number of edges. What is this number? Prove your answer.
- For $n \geq 5$, the degree sequence of all possible eight-graphs is the same. Let G be an eight-graph with n vertices. How many vertices in G have degree 0? How many vertices in G have degree 1? How many vertices in G have degree 2? How many vertices in G have degree 3? How many vertices in G have degree 4? How many vertices in G have degree greater than 4? Justify your answers.

16. Illustrate the six possible molecules of the type C_4H_y for some positive integer y in which a connection between two C -atoms can have at most one edge, the degree of each one of the C -atoms is 4, and the degree of each one of the H -atoms is 1.

Hint: The number of the H -atoms y is uniquely determined by the inter-connections structure among the four C -atoms.

17. Let G be a simple (no parallel edges and no self loops) undirected graph on the $n \geq 1$ vertices $\{v_1, v_2, \dots, v_n\}$.

If vertex v_i has d_i neighbors (equivalently, the degree of v_i is d_i), then the **degree sequence** of G is $D_G = (d_1, d_2, \dots, d_n)$. If a sequence D has a graph G for which D is G 's degree sequence then D is a **graphic** sequence and G is a **realization** of D .

One of the following three degree sequences is not a graphic sequence, one of them is a graphic sequence with exactly one realization, and one of them is a graphic sequence with two non-isomorphic realizations. Which is which?

If a sequence is not graphic explain why. If a sequence is graphic, illustrate all of its possible realizations.

- (a) $D = (4, 4, 2, 2, 2)$
- (b) $D = (4, 4, 1, 1, 1, 1)$
- (c) $D = (2, 2, 2, 2, 2, 2)$

18. The sequence $S = (5, 4, 3, 3, 2, 1, 1, 1)$ is graphic. Associate the eight vertices A, B, C, D, E, F, G, H with the degrees by their non-increasing order.

- (a) Construct the realization graph of the sequence S by running the algorithm in which the pivot is always the vertex with the current highest degree. When there is more than one candidate for the pivot or when there are several choices for selecting the neighbors for the pivot, give preference to the vertices which appear first in the alphabetical order from A to H .

Show all the recursive sequences until you reach the final $(0, 0, 0, 0, 0, 0, 0, 0)$ sequence. But illustrate only the final realization graph of the original sequence.

- (b) Construct the realization graph of the sequence S by running the algorithm in which the pivot is always the vertex with the current lowest degree. When there are more than one candidate for the pivot, give preference to the vertex which appear last in the alphabetical order from A to H . However, when there are several choices for selecting the neighbors for a pivot, give preference to the vertices which appear first in the alphabetical order from A to H .

Show all the recursive sequences until you reach the final $(0, 0, 0, 0, 0, 0, 0, 0)$ sequence. But illustrate only the final realization graph of the original sequence.

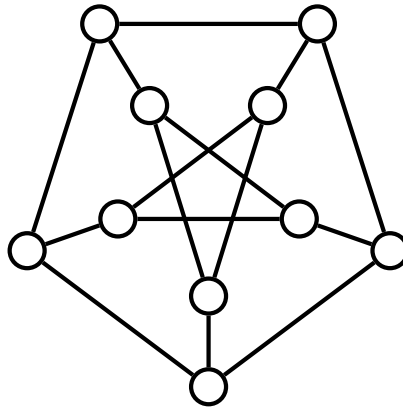
- (c) Prove that the two realizations from parts (a) and (b) are non-isomorphic.
- (d) Find another sequence of pivots (recall that in any recursive round, any vertex may be the pivot as long as it is connected to the current highest degree vertices) that generates a third non-isomorphic realization graph of the sequence S .

Show all the recursive sequences until you reach the final $(0, 0, 0, 0, 0, 0, 0, 0)$ sequence. But illustrate only the final realization graph of the original sequence.

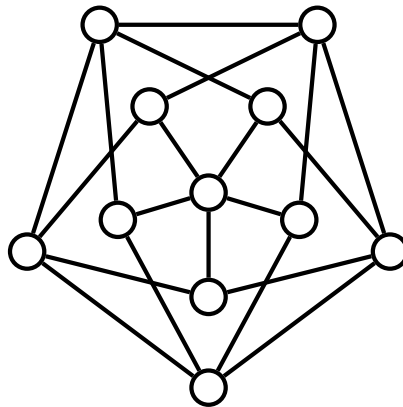
- (e) Prove that the third realization is not isomorphic to the realizations from parts (a) and (b).

19. A coloring of a graph is an assignment of “colors” to the vertices such that the colors assigned to the two vertices of any edge are different.

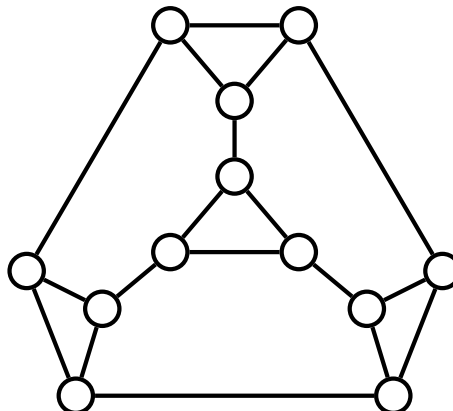
- (a) How many colors are needed to color a null graph with n vertices? Explain.
- (b) How many colors are needed to color a complete graph with n vertices? Explain.
- (c) How many colors are needed to color a tree with n vertices? Explain.
- (d) How many colors are needed to color a bipartite graph? Explain.
- (e) How many colors are needed to color a cycle graph with n vertices? Explain. Note that the answer is different when n is even and when n is odd.
- (f) Color the Petersen graph with as few colors as possible. Explain why it is impossible to color it with fewer colors.



- (g) Color the Grötzsch graph with as few colors as possible. Explain why it is impossible to color it with fewer colors.



- (h) Color the FourTriangles graph with as few colors as possible. Explain why it is impossible to color it with fewer colors.



20. Let G be a simple undirected graph with $n \geq 1$ vertices. An **independent set** in G is a set of vertices I that have no edges among themselves. That is, for any pair of vertices $u, v \in I$, the graph G does not include the edge (u, v) .

Observe that the set of all the vertices of the null graph is an independent set of size n because the graph has no edges while in the complete graph only one vertex can be a member in an independent set because there is an edge between any pair of vertices.

What is the size of the **largest** independent set in the Petersen graph? Give an example of such a set and prove that a larger set does not exist.

21. Let G be a simple undirected graph with $n \geq 1$ vertices. A **clique** in G is a set of vertices C that contains all the possible edges among themselves. That is, for any pair of vertices $u, v \in C$, the graph G includes the edge (u, v) .

Observe that the set of all the vertices of the complete graph is a clique of size n because there is an edge between any pair of vertices while in the null graph only one vertex can be a member in a clique because the graph has no edges.

What is the size of the **largest** clique in the Petersen graph? Give an example of such a clique and prove that a larger set does not exist.

22. Let G be a simple undirected graph with $n \geq 1$ vertices. A **vertex cover** in G is a set of vertices VC for which each edge has at least one of its vertices in the set. That is, for any edge (u, v) of G either $u \in VC$ or $v \in VC$.

Observe that the empty set could be a vertex cover of the null graph because there are no edges to cover. However, in the complete graph any vertex cover must contain all the vertices except one because if the vertex cover misses two vertices then there would be no vertex in the set to cover this edge.

What is the size of the **smallest** vertex cover in the Petersen graph? Give an example of such a set and prove that a smaller set does not exist.

23. Let G be a simple undirected graph with $n \geq 1$ vertices. A **dominating set** in G is a set of vertices D for which all other vertices have at least one neighbor in D . That is, for any vertex v of G either $v \in D$ or there exists a neighbor U of v such that $u \in D$.

Observe that in the null graph any dominating set must contain all the vertices and therefore this is the only possible dominating set. On the other hand, in the complete graph any vertex dominates itself and the rest of the vertices and therefore it can be a dominating set by itself.

What is the size of the **smallest** dominating set in the Petersen graph? Give an example of such a set and prove that a smaller set does not exist.

24. A **path** $\mathcal{P} = \langle v_0, v_1, \dots, v_k \rangle$ of length k (number of edges) is an ordered list of $k + 1$ vertices such that the edge (v_i, v_{i+1}) exists for $0 \leq i \leq k - 1$ and all the edges are different. A **cycle** $\mathcal{C} = \langle v_0, v_1, \dots, v_{k-1}, v_0 \rangle$ of length k is a path that starts and ends with the same vertex.

An **Euler path** is a path that contains all the edges in the graph. An **Euler cycle** is a cycle that contains all the edges in the graph. The Petersen graph has neither an Euler path nor an Euler cycle.

- (a) Find one of the longest paths (does not have to be simple) in the Petersen graph.
- (b) Find one of the longest cycles (does not have to be simple) in the Petersen graph.

25. A **path** $\mathcal{P} = \langle v_0, v_1, \dots, v_k \rangle$ of length k (number of edges) is an ordered list of $k + 1$ vertices such that the edge (v_i, v_{i+1}) exists for $0 \leq i \leq k - 1$ and all the edges are different. A **cycle** $\mathcal{C} = \langle v_0, v_1, \dots, v_{k-1}, v_0 \rangle$ of length k is a path that starts and ends with the same vertex. In a **simple path** all the vertices are different. In a **simple cycle** all the vertices except $v_0 = v_k$ are different.

A **Hamiltonian path** is a simple path that contains all the vertices in the graph. A **Hamiltonian cycle** is a simple cycle that contains all the vertices in the graph. The Petersen graph has a Hamiltonian path but does not have a Hamiltonian cycle.

- (a) Find one of the Hamiltonian paths in the Petersen graph.
- (b) Find one of the longest simple cycles in the Petersen graph.

26. Trees and bipartite graphs.

- (a) Prove or disprove: every tree is a bipartite graph.
- (b) Prove or disprove: every bipartite graph is a tree.

27. Prove that for every $n \geq 5$ there exists a graph with n vertices, all of which have degree 4.

28. Prove that no graph has all the degrees different. That is, prove that in a degree sequence there is at least one repeated number.

29. A *bridge* in a connected graph is an edge (u, v) whose removal from the graph disconnects it.

Prove that if G is a connected graph for which all the vertices have an even degree, then G does not have a bridge.