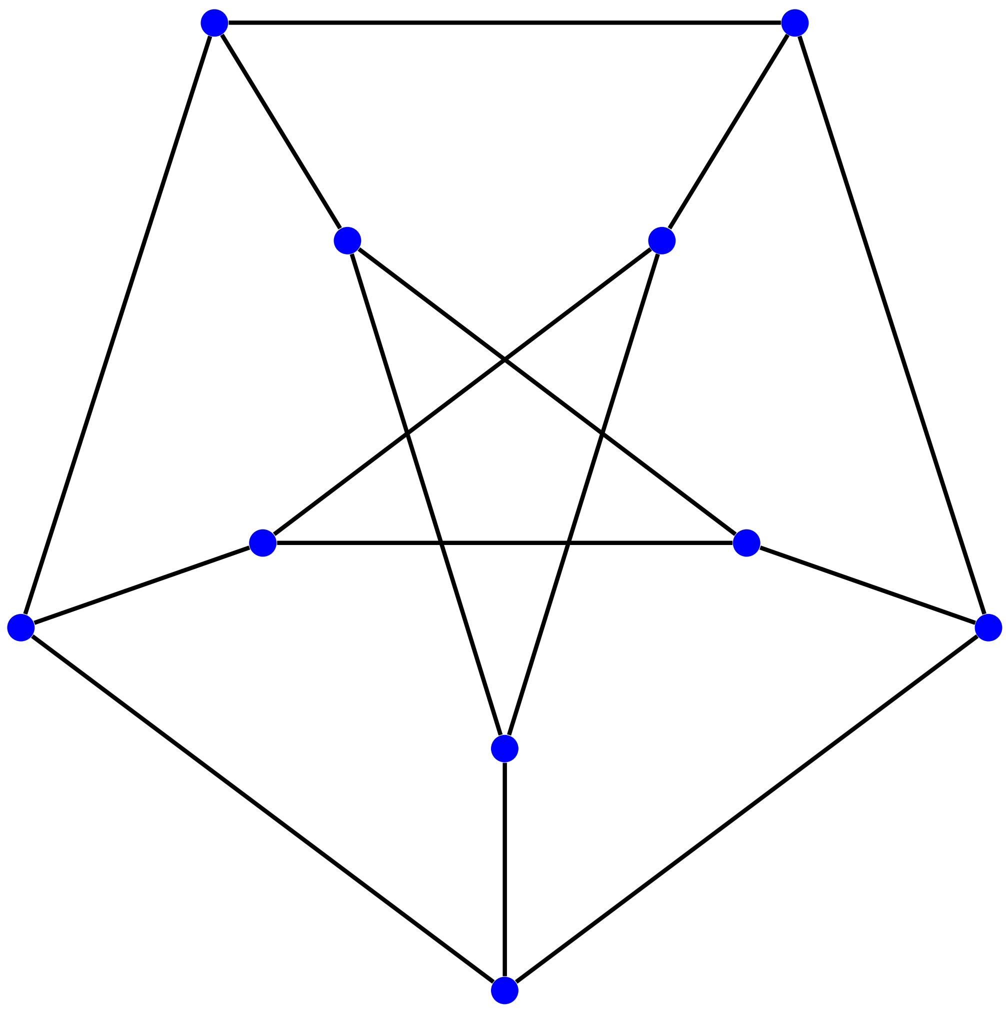
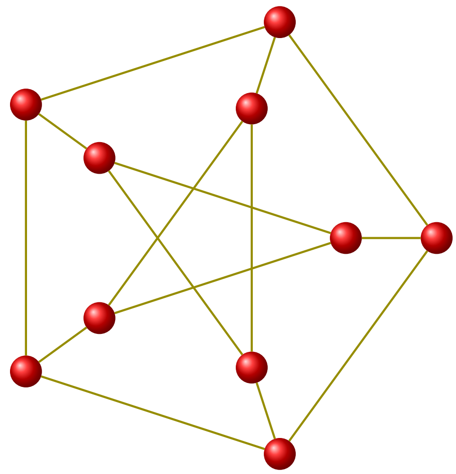
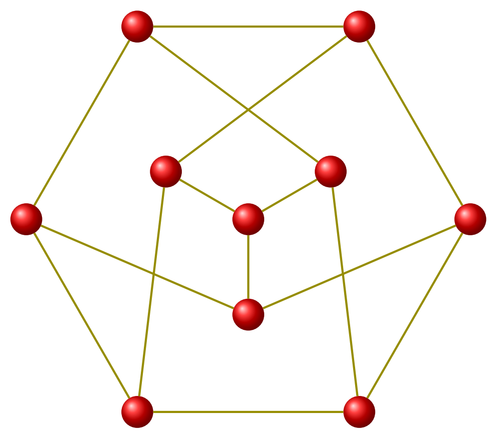
Discrete Structures

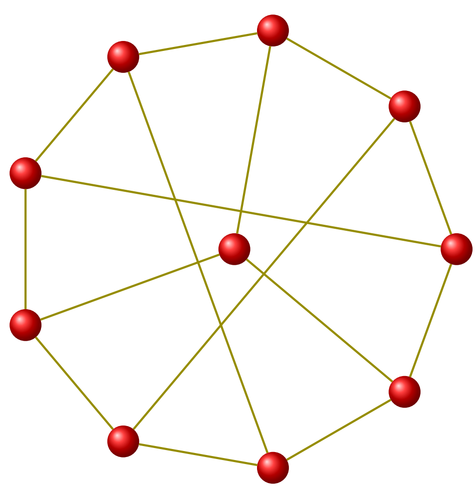
Graphs Practice Problems: Solutions



1. Three symmetric ("nice") drawings of the Petersen graph:

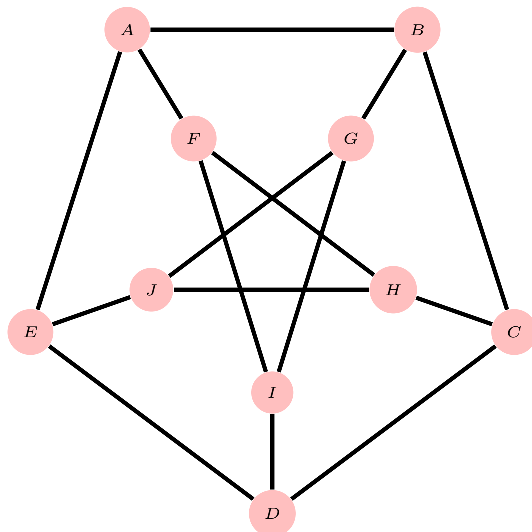






Additional drawings: `http://mathworld.wolfram.com/PetersenGraph.html`

2.



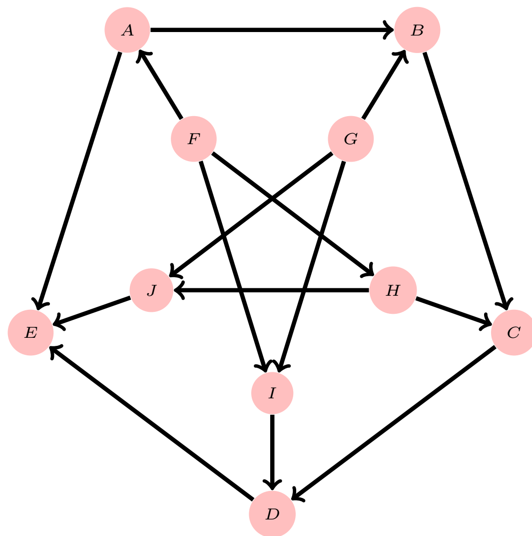
(a) The adjacency matrix of the above undirected Petersen graph.

	A	B	C	D	E	F	G	H	I	J
A	0	1	0	0	1	1	0	0	0	0
B	1	0	1	0	0	0	1	0	0	0
C	0	1	0	1	0	0	0	1	0	0
D	0	0	1	0	1	0	0	0	1	0
E	1	0	0	1	0	0	0	0	0	1
F	1	0	0	0	0	0	0	1	1	0
G	0	1	0	0	0	0	0	0	1	1
H	0	0	1	0	0	1	0	0	0	1
I	0	0	0	1	0	1	1	0	0	0
J	0	0	0	0	1	0	1	1	0	0

(b) The adjacency lists of the above undirected Petersen graph.

$$\begin{array}{ccc} A & \rightarrow & (B, E, F) \\ B & \rightarrow & (A, C, G) \\ C & \rightarrow & (B, D, H) \\ D & \rightarrow & (C, E, I) \\ E & \rightarrow & (A, D, J) \\ F & \rightarrow & (A, H, I) \\ G & \rightarrow & (B, I, J) \\ H & \rightarrow & (C, F, J) \\ I & \rightarrow & (D, F, G) \\ J & \rightarrow & (E, G, H) \end{array}$$

3.



(a) The adjacency matrix of the above directed Petersen graph.

	A	B	C	D	E	F	G	H	I	J
A	0	1	0	0	1	0	0	0	0	0
B	0	0	1	0	0	0	0	0	0	0
C	0	0	0	1	0	0	0	0	0	0
D	0	0	0	0	1	0	0	0	0	0
E	0	0	0	0	0	0	0	0	0	0
F	1	0	0	0	0	0	0	1	1	0
G	0	1	0	0	0	0	0	0	1	1
H	0	0	1	0	0	0	0	0	0	1
I	0	0	0	1	0	0	0	0	0	0
J	0	0	0	0	1	0	0	0	0	0

(b) The incoming and outgoing adjacency lists of the above directed Petersen graph.

$$\begin{array}{rcccl}
(F) & \rightarrow & A & \rightarrow & (B, E) \\
(A, G) & \rightarrow & B & \rightarrow & (C) \\
(B, H) & \rightarrow & C & \rightarrow & (D) \\
(C, I) & \rightarrow & D & \rightarrow & (E) \\
(A, D, J) & \rightarrow & E & \rightarrow & () \\
() & \rightarrow & F & \rightarrow & (A, H, I) \\
() & \rightarrow & G & \rightarrow & (B, I, J) \\
(F) & \rightarrow & H & \rightarrow & (C, J) \\
(F, G) & \rightarrow & I & \rightarrow & (D) \\
(G, H) & \rightarrow & J & \rightarrow & (E)
\end{array}$$

4. A simple undirected **labeled** graph H is a **labeled-sub-graph** of another simple undirected **labeled** graph G if both G and H have the same set of vertices and every edge of H is also an edge of G (however there could be edges of G that are not edges of H).

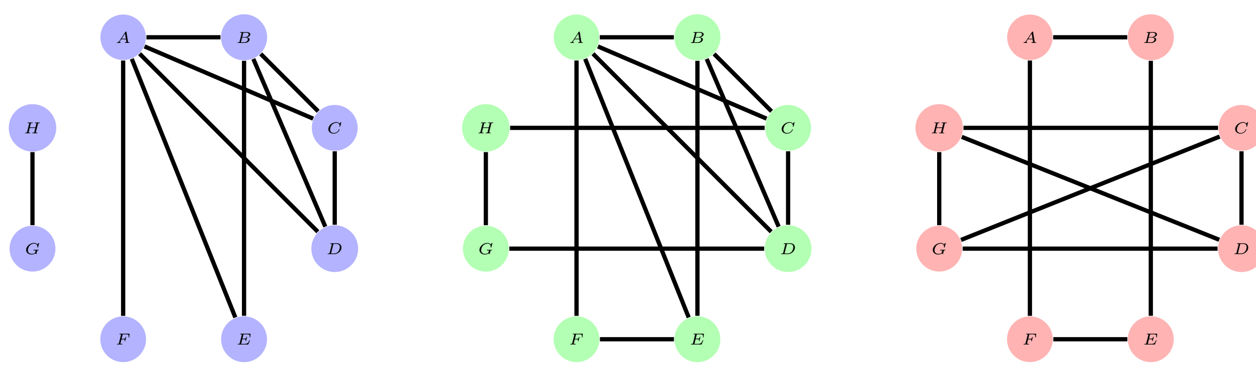
   For $n \geq 1$, let G and H be two simple undirected labeled graphs on the vertices $\{1, 2, \ldots, n\}$. Assume that both graphs are represented by adjacency matrices.

   Describe how to determine if H is a labeled-sub-graph of G.

   Answer: Let P be the adjacency matrix of G and let Q be the adjacency matrix of H. Compare all the $n \times n$ entries in Q with the corresponding entries in P. It follows that H is a labeled-sub-graph of G if and only if $Q(i, j) \leq P(i, j)$ for all $1 \leq i, j \leq n$. In other words, H is a labeled-sub-graph of G only if $P(i, j) = 1$ for all $1 \leq i, j \leq n$ such that $Q(i, j) = 1$.

   Example: The left (blue) graph is a labeled-sub-graph of the middle (green) graph because all the edges in the blue graph appear in the green graph while the right (red) graph is not a labeled-sub-graph of the green graph because it includes the edges (C, G) and (D, H) which are not edges in the green graph.

   The bottom adjacency matrix corresponding to the green graph has a 1 in any entry in which the left adjacency matrix corresponding to the blue graph has a 1 entry. However, both entries (C, G) and (D, H) in the right adjacency matrix corresponding to the red graph has a 1 entry while in the bottom matrix these entries are 0.



	A	B	C	D	E	F	G	H
A	0	1	1	1	1	1	0	0
B	1	0	1	1	1	0	0	0
C	1	1	0	1	0	0	0	0
D	1	1	1	0	0	0	0	0
E	1	1	0	0	0	0	0	0
F	1	0	0	0	0	0	0	0
G	0	0	0	0	0	0	0	1
H	0	0	0	0	0	0	1	0

	A	B	C	D	E	F	G	H
A	0	1	0	0	0	1	0	0
B	1	0	0	0	1	0	0	0
C	0	0	0	1	0	0	1	1
D	0	0	1	0	0	0	1	1
E	0	1	0	0	0	1	0	0
F	1	0	0	0	1	0	0	0
G	0	0	0	1	0	0	0	1
H	0	0	1	0	0	0	1	0

	A	B	C	D	E	F	G	H
A	0	1	1	1	1	1	0	0
B	1	0	1	1	1	0	0	0
C	1	1	0	1	0	0	0	1
D	1	1	1	0	0	0	1	0
E	1	1	0	0	0	1	0	0
F	1	0	0	0	1	0	0	0
G	0	0	0	1	0	0	0	1
H	0	0	1	0	0	0	1	0

5. The **union** graph of two simple undirected labeled graphs with the same set of vertices is a simple undirected labeled graph with the same vertex set that contains all of the edges that belong to at least one of the graphs.

For $n \geq 1$, let G and H be two simple undirected labeled graphs on the vertices $\{1, 2, \ldots, n\}$. Assume that these graphs are represented by adjacency matrices.

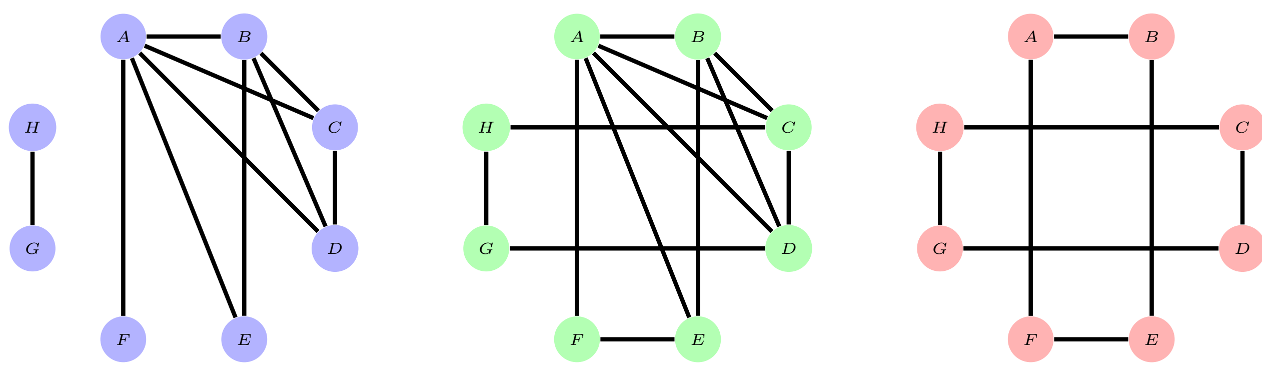
Describe how to construct the adjacency matrix of the union graph of G and H.

Answer: Let P be the adjacency matrix of G and let Q be the adjacency matrix of H. Construct R the adjacency matrix of $U = G \cup H$ as follows. For all $1 \leq i, j \leq n$:

$$R(i, j) = \max\{P(i, j), Q(i, j)\}$$

For all $1 \leq i, j \leq n$, it follows that $R(i, j) = 1$ iff either $P(i, j) = 1$ or $Q(i, j) = 1$ or equivalently, $R(i, j) = 0$ iff both $P(i, j) = 0$ and $Q(i, j) = 0$.

Example: The middle (green) graph is the union of the left (blue) graph with the right (red) graph. The bottom adjacency matrix corresponding to the green graph was generated with the **max** operator applied to all the $64 = 8 \times 8$ entries of the top left adjacency matrix corresponding to the blue graph with the top right adjacency matrix corresponding to the red graph.



	A	B	C	D	E	F	G	H
A	0	1	1	1	1	1	0	0
B	1	0	1	1	1	0	0	0
C	1	1	0	1	0	0	0	0
D	1	1	1	0	0	0	0	0
E	1	1	0	0	0	0	0	0
F	1	0	0	0	0	0	0	0
G	0	0	0	0	0	0	0	1
H	0	0	0	0	0	0	1	0

	A	B	C	D	E	F	G	H
A	0	1	0	0	0	1	0	0
B	1	0	0	0	1	0	0	0
C	0	0	0	1	0	0	0	1
D	0	0	1	0	0	0	1	0
E	0	1	0	0	0	1	0	0
F	1	0	0	0	1	0	0	0
G	0	0	0	1	0	0	0	1
H	0	0	1	0	0	0	1	0

	A	B	C	D	E	F	G	H
A	0	1	1	1	1	1	0	0
B	1	0	1	1	1	0	0	0
C	1	1	0	1	0	0	0	1
D	1	1	1	0	0	0	1	0
E	1	1	0	0	0	1	0	0
F	1	0	0	0	1	0	0	0
G	0	0	0	1	0	0	0	1
H	0	0	1	0	0	0	1	0

6. The **intersection** graph of two simple undirected labeled graphs with the same set of vertices is a simple undirected labeled graph with the same vertex set that contains all of the edges that belong to both graphs.

For $n \geq 1$, let G and H be two simple undirected labeled graphs on the vertices $\{1, 2, \ldots, n\}$. Assume that these graphs are represented by adjacency matrices.

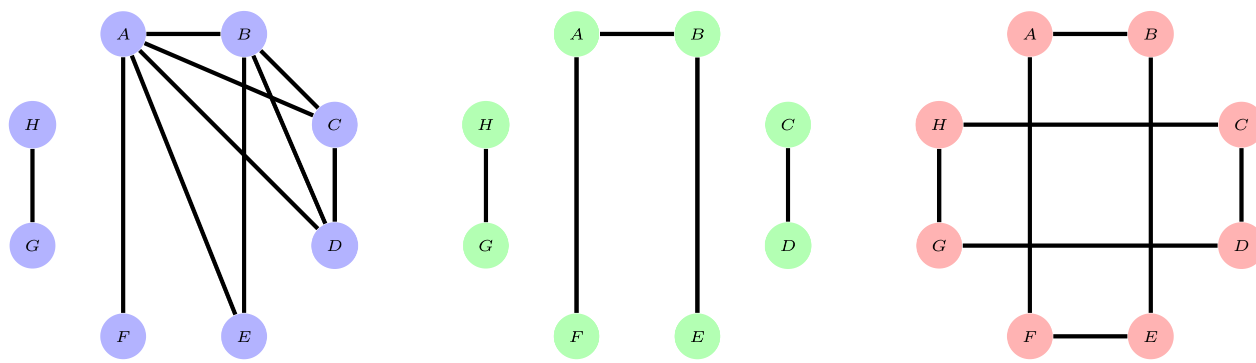
Describe how to construct the adjacency matrix of the intersection graph of G and H.

Answer: Let P be the adjacency matrix of G and let Q be the adjacency matrix of H. Construct R the adjacency matrix of $I = G \cap H$ as follows. For all $1 \leq i, j \leq n$:

$$R(i, j) = \min \{P(i, j), Q(i, j)\}$$

For all $1 \leq i, j \leq n$, it follows that $R(i, j) = 1$ iff both $P(i, j) = 1$ and $Q(i, j) = 1$ or equivalently, $R(i, j) = 0$ iff either $P(i, j) = 0$ or $Q(i, j) = 0$.

Example: The graph in the middle (green) is the intersection of the graph in the left (blue) with the graph in the right (red). The bottom adjacency matrix corresponding to the green graph was generated with the **min** operator applied to all the $64 = 8 \times 8$ entries of the top left adjacency matrix corresponding to the blue graph with the top right adjacency matrix corresponding to the red graph.



	A	B	C	D	E	F	G	H
A	0	1	1	1	1	1	0	0
B	1	0	1	1	1	0	0	0
C	1	1	0	1	0	0	0	0
D	1	1	1	0	0	0	0	0
E	1	1	0	0	0	0	0	0
F	1	0	0	0	0	0	0	0
G	0	0	0	0	0	0	0	1
H	0	0	0	0	0	0	1	0

	A	B	C	D	E	F	G	H
A	0	1	0	0	0	1	0	0
B	1	0	0	0	1	0	0	0
C	0	0	0	1	0	0	0	1
D	0	0	1	0	0	0	1	0
E	0	1	0	0	0	1	0	0
F	1	0	0	0	1	0	0	0
G	0	0	0	1	0	0	0	1
H	0	0	1	0	0	0	1	0

	A	B	C	D	E	F	G	H
A	0	1	0	0	0	1	0	0
B	1	0	0	0	1	0	0	0
C	0	0	0	1	0	0	0	0
D	0	0	1	0	0	0	0	0
E	0	1	0	0	0	0	0	0
F	1	0	0	0	0	0	0	0
G	0	0	0	0	0	0	0	1
H	0	0	0	0	0	0	1	0

7. Two simple (no self loops and no parallel edges) undirected graphs G and H are **isomorphic** if there exists a one-to-one function f from the vertices of G to the vertices of H such that an edge (u, v) exists in G **if and only if** the edge $(f(u), f(v))$ exists in H.

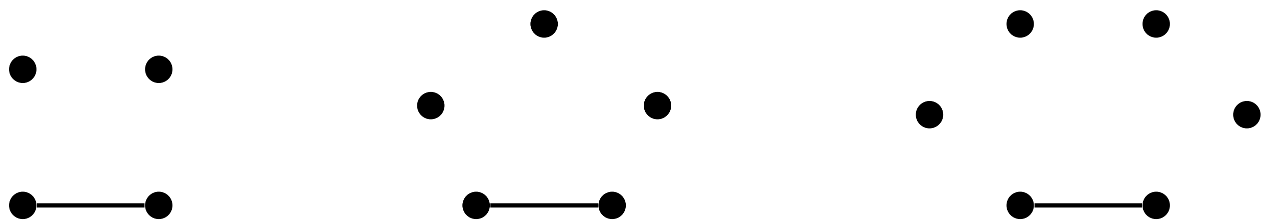
   For each one of the following four parts determine if G and H are (i) always isomorphic, or (ii) could never be isomorphic, or (iii) sometimes isomorphic and sometimes not isomorphic.

   (a) G and H have the same number of edges. However, G has n vertices while H has $n+1$ vertices for $n \geq 1$.

   Answer: G and H can never be isomorphic because two isomorphic graphs must have the same number of vertices.

   (b) Both graphs have $n \geq 2$ vertices and exactly one edge.

   Answer: Always isomorphic because for $n \geq 2$ there is only one isomorphic graph with n vertices and 1 edge. See below these graphs for $n = 4, 5, 6$.

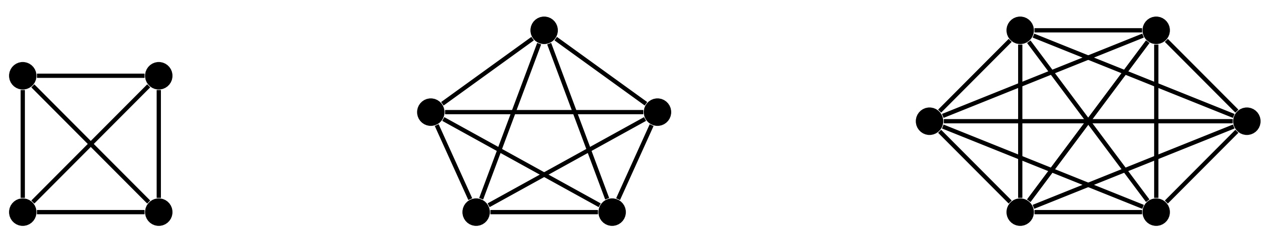


   (c) Both graphs have $n \geq 4$ vertices and exactly two edges.

   Answer: Sometime isomorphic and sometimes not isomorphic because there are two non-isomorphic graphs with n vertices and 2 edges. In one of them the two edges intersect and in the other the two edges are disjoint.

   (d) Both graphs have $n \geq 2$ vertices and all possible edges exist.

   Answer: Always isomorphic because there is only one isomorphic graph with $n \geq 2$ vertices and all the $\frac{n(n-1)}{2}$ possible edges. See below these graphs for $n = 4, 5, 6$.



8. Let G and H be two simple undirected graphs, each with 6 vertices and the **same** number of edges.

   In each of the following parts, if it is possible to find two non-isomorphic graphs G and H that satisfy the conditions, draw examples for G and H and explain why G and H are not isomorphic. If it is not possible, explain why it is not possible to find two such graphs.

   (a) G and H are not isomorphic.

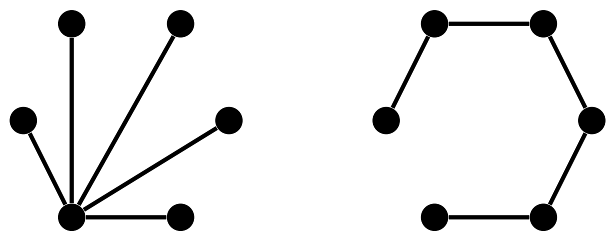
   (b) G and H are not isomorphic and every vertex has degree 5.

   (c) G and H are not isomorphic and every vertex has degree 2.

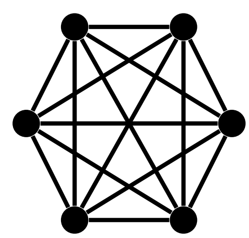
   (d) G and H are not isomorphic and every vertex has degree 1.

Answers:

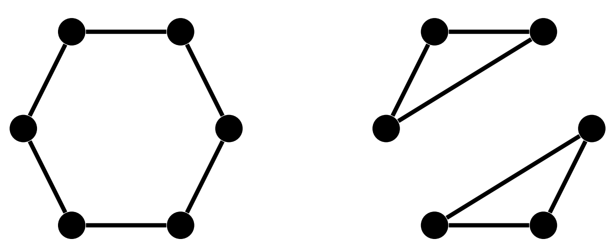
(a) The following two graphs are not isomorphic although both have 6 vertices and 5 edges. Both graphs are trees but one of them is a star while the other is a path.



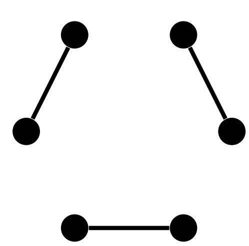
(b) The complete graph with 6 vertices and all 15 edges is the only non-isomorphic graph with 6 vertices in which every vertex has degree 5.



(c) There are two non-isomorphic graphs with 6 vertices and 6 edges in which every vertex has degree 2. One of them is a cycle of size 6 while the other contains two triangles (cycles of size 3).

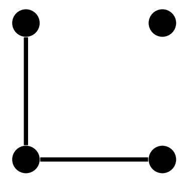


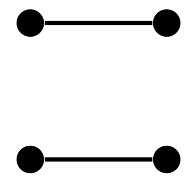
(d) The following is the only non-isomorphic graph with 6 vertices in which every vertex has degree 1.



9. In the following questions explain why the two graphs you drew are not isomorphic.

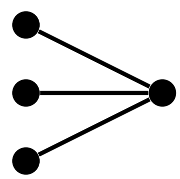
(a) Draw two non-isomorphic simple graphs with 4 vertices and 2 edges.

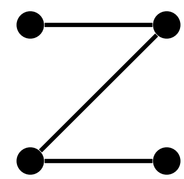




The above two graphs each has four vertices and two edges. The two graphs are not isomorphic because in one of them the two edges intersect and in one of them the two edges are disjoint.

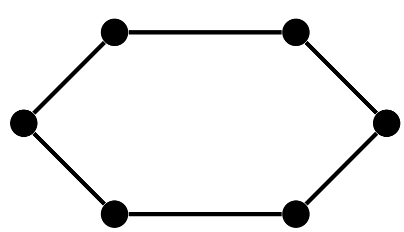
(b) Draw two non-isomorphic simple bipartite graphs with 4 vertices and 3 edges.

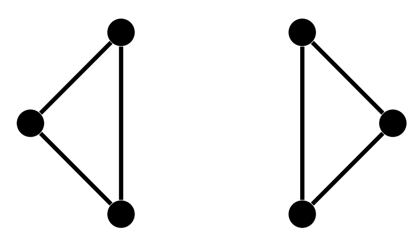




The above two bipartite graphs each has four vertices and three edges. The two graphs are not isomorphic because one of them has a vertex with degree 3 while in the other the maximum degree is 2.

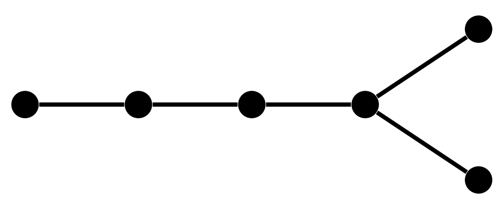
(c) Draw two non-isomorphic simple graphs with 6 vertices and 6 edges for which the degrees of all the vertices are 2.

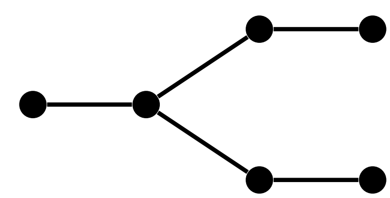




The above two graphs each has six vertices and six edges and the same degree sequence $(2, 2, 2, 2, 2, 2)$. The two graphs are not isomorphic because one of them has one cycle of length 6 and one of them has two cycles of length 3.

(d) Draw two non-isomorphic simple trees with 6 vertices and 3 leaves (vertices whose degree is 1).

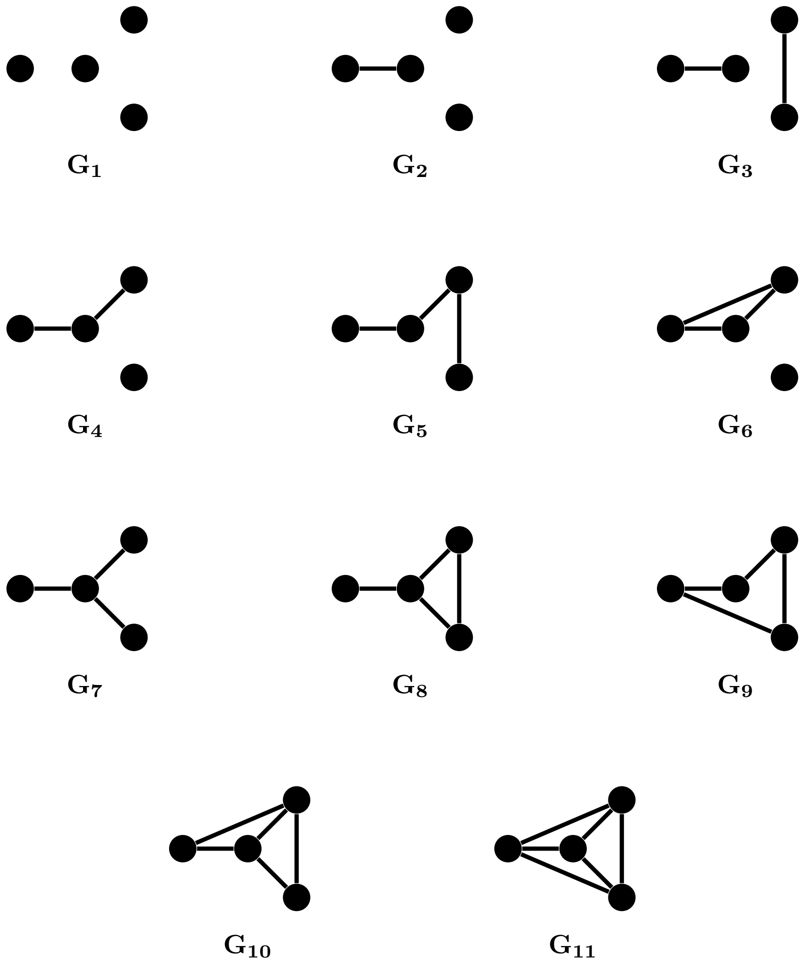




The above two graphs each has six vertices and five edges. Both graphs are trees with three leaves whose degree sequences are $(3, 2, 2, 1, 1, 1)$. These two trees are not isomorphic because in one of them the vertex of degree 3 is connected to two leaves and in one of them the vertex of degree 3 is connected to only one leaf.

10. Below are all the 11 non-isomorphic graphs with 4 vertices.
   (a) Identify all the graphs that are connected.
   (b) Identify all the graphs that are forests.
   (c) Identify all the graphs that are trees.
   (d) Identify all the graphs that are regular (all vertices have the same degree).
   (e) Identify all the graphs that are bipartite.

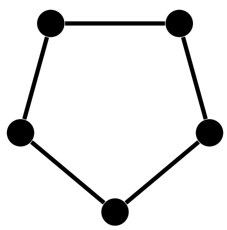
   Fill the table below with Yes (Y) and No (N).



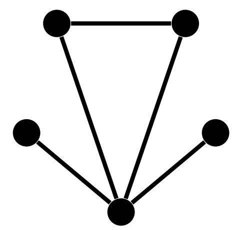
	G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_8	G_9	G_{10}	G_{11}
connected graphs	**No**	**No**	**No**	**No**	**Yes**	**No**	**Yes**	**Yes**	**Yes**	**Yes**	**Yes**
forests	**Yes**	**Yes**	**Yes**	**Yes**	**Yes**	**No**	**Yes**	**No**	**No**	**No**	**No**
trees	**No**	**No**	**No**	**No**	**Yes**	**No**	**Yes**	**No**	**No**	**No**	**No**
regular graphs	**Yes**	**No**	**Yes**	**No**	**No**	**No**	**No**	**No**	**Yes**	**No**	**Yes**
bipartite	**Yes**	**Yes**	**Yes**	**yes**	**Yes**	**No**	**Yes**	**No**	**Yes**	**No**	**No**

11. Below are all the 6 non-isomorphic graphs with 5 vertices and 5 edges.
    (a) Identify all the graphs that are connected.
    (b) Identify all the graphs that are regular (all vertices have the same degree).
    (c) Identify all the graphs that are bipartite.

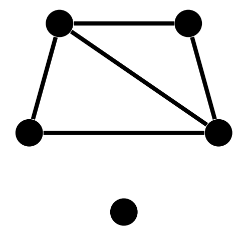
Fill the table below with Yes (Y) and No (N).



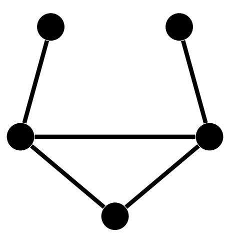
$\mathbf{H_1}$



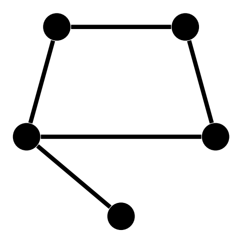
$\mathbf{H_2}$



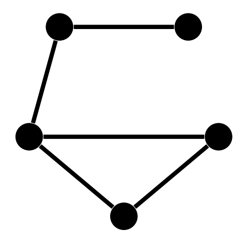
$\mathbf{H_3}$



$\mathbf{H_4}$



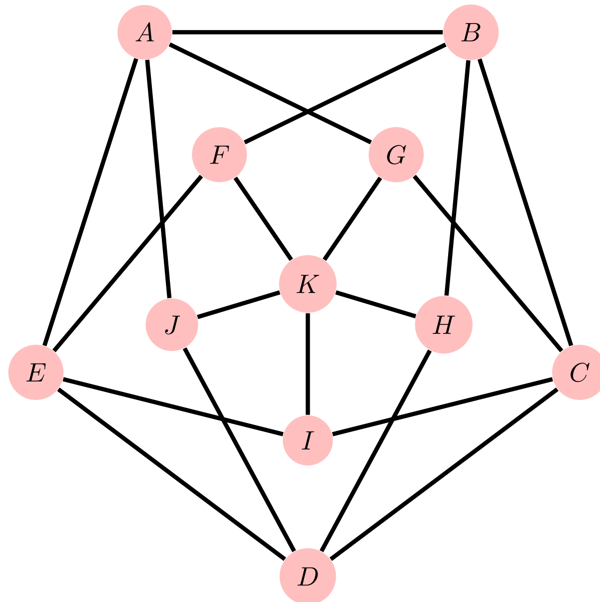
$\mathbf{H_5}$



$\mathbf{H_6}$

	H_1	H_2	H_3	H_4	H_5	H_6
connected graphs	**Yes**	**Yes**	**No**	**Yes**	**Yes**	**Yes**
regular graphs	**Yes**	**No**	**No**	**No**	**No**	**No**
bipartite	**No**	**No**	**No**	**No**	**Yes**	**No**

12. The following is the Grötzsch graph with the 11 vertices $A, B, C, D, E, F, G, H, I, J, K$.



(a) The adjacency matrix of the Grötzsch graph.

	A	B	C	D	E	F	G	H	I	J	K
A	0	1	0	0	1	0	1	0	0	1	0
B	1	0	1	0	0	1	0	1	0	0	0
C	0	1	0	1	0	0	1	0	1	0	0
D	0	0	1	0	1	0	0	1	0	1	0
E	1	0	0	1	0	1	0	0	1	0	0
F	0	1	0	0	1	0	0	0	0	0	1
G	1	0	1	0	0	0	0	0	0	0	1
H	0	1	0	1	0	0	0	0	0	0	1
I	0	0	1	0	1	0	0	0	0	0	1
J	1	0	0	1	0	0	0	0	0	0	1
K	0	0	0	0	0	1	1	1	1	1	0

(b) The adjacency list of the Grötzsch graph:

$$
\begin{aligned}
A &\rightarrow (B, E, G, J) \\
B &\rightarrow (A, C, F, H) \\
C &\rightarrow (B, D, G, I) \\
D &\rightarrow (C, E, H, J) \\
E &\rightarrow (A, D, F, I) \\
F &\rightarrow (B, E, K) \\
G &\rightarrow (A, C, K) \\
H &\rightarrow (B, D, K) \\
I &\rightarrow (C, E, K) \\
J &\rightarrow (A, D, K) \\
K &\rightarrow (F, G, H, I, J)
\end{aligned}
$$

(c) Is the Grötzsch graph connected?

Answer: Yes. The Grötzsch graph is connected because it is possible to verify that there is a path between any pair of vertices. One way to show this is to list all the paths from the vertices $A, B, \ldots J$ to K. Then for any two non-neighbors vertices or vertices without a common neighbor $X, Y \in \{A, B, \ldots J\}$, a path from X to Y can go from X to K and then from K to Y.

In fact, one can prove a stronger claim that there is a path of length at most 2 between any pair of vertices. This can be shown for all possible $55 = \binom{11}{2}$ pairs. Using symmetry, the case analysis does not have to be long.

(d) Is the Grötzsch graph a tree?

Answer: No. Because trees do not have cycles and (A, B, F, E, A) is one of the cycles in the Grötzsch graph. Also trees with 11 vertices have exactly 10 edges while the Grötzsch graph has 20 edges.

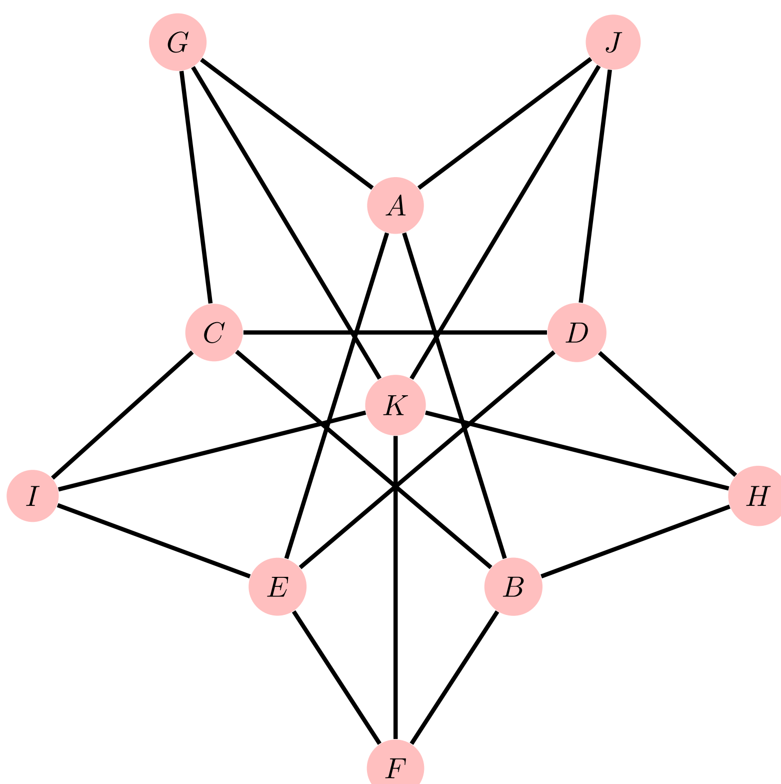
(e) Is the Grötzsch graph a bipartite graph?

Answer: No. Because bipartite graphs do not have cycles of size 5 and (A, B, C, D, E, A) is a cycle in the Grötzsch graph.

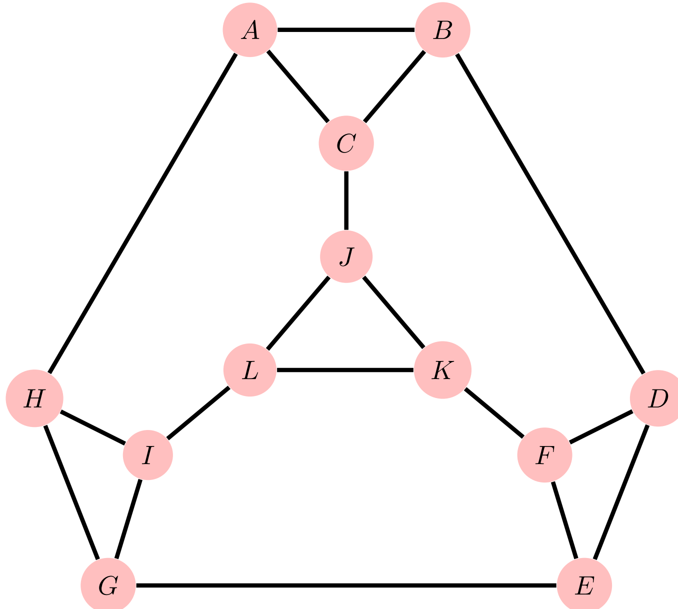
The claims about cycles of size 5 can be proven as follows. If A is on one side of the bipartite graph, then its neighbors B and E must be on the other side of the bipartite graph. Then, their neighbors C and D must be on the same side of the bipartite graph as A. This is a contradiction, because there is an edge between D and E and therefore they cannot be on the same side of the bipartite graph.

The above proof can be generalized to any cycle of an odd size proving that a cycle of an odd size is not a bipartite graph.

(f) The following assignment of the 11 labels $A, B, C, D, E, F, G, H, I, J, K$ to the 11 vertices in the drawing below demonstrates that this drawing is isomorphic to the original drawing of the Grötzsch graph. That is, for any pair of vertices $u, v \in \{A, B, C, D, E, F, G, H, I, J, K\}$, the edge (u, v) exists in the original drawing if and only if it exists in the drawing below.



13. The following is the FourTriangles graph with the 12 vertices $A, B, C, D, E, F, G, H, I, J, K, L$.



(a) The adjacency matrix of the FourTriangles graph.

	A	B	C	D	E	F	G	H	I	J	K	L
A	0	1	1	0	0	0	0	1	0	0	0	0
B	1	0	1	1	0	0	0	0	0	0	0	0
C	1	1	0	0	0	0	0	0	0	1	0	0
D	0	1	0	0	1	1	0	0	0	0	0	0
E	0	0	0	1	0	1	1	0	0	0	0	0
F	0	0	0	1	1	0	0	0	0	0	1	0
G	0	0	0	0	1	0	0	1	1	0	0	0
H	1	0	0	0	0	0	1	0	1	0	0	0
I	0	0	0	0	0	0	1	1	0	0	0	1
J	0	0	1	0	0	0	0	0	0	0	1	1
K	0	0	0	0	0	1	0	0	0	1	0	1
L	0	0	0	0	0	0	0	0	1	1	1	0

(b) The adjacency list of the FourTriangles graph:

$$\begin{array}{lcl} A & \rightarrow & (B, C, H) \\ B & \rightarrow & (A, C, D) \\ C & \rightarrow & (A, B, J) \\ D & \rightarrow & (B, E, F) \\ E & \rightarrow & (D, F, G) \\ F & \rightarrow & (D, E, K) \\ G & \rightarrow & (E, H, I) \\ H & \rightarrow & (A, G, I) \\ I & \rightarrow & (G, H, L) \\ J & \rightarrow & (C, K, L) \\ K & \rightarrow & (F, J, L) \\ L & \rightarrow & (I, J, K) \end{array}$$

(c) Is the FourTriangles graph connected?

Answer: Yes. The FourTriangles graph is connected because it is possible to verify that there is a path between any pair of vertices. In fact, one can prove a stronger claim that there is a path of length at most 3 between any pair of vertices.

Assume that X and Y are two different vertices. If they belong to the same triangle then there is a path of length 1 between them. Otherwise, with one edge or with a path of two edges, it is possible to go from X to the triangle that contains Y. Finally, with at most one additional edge, this edge or a path can be extended to a path that reaches Y.

(d) Is the FourTriangles graph a tree?

Answer: No. Because trees do not have cycles and the FourTriangles graph has four triangles and several larger size cycles. Also trees with 12 vertices have exactly 11 edges while the FourTriangles graph has 18 edges.

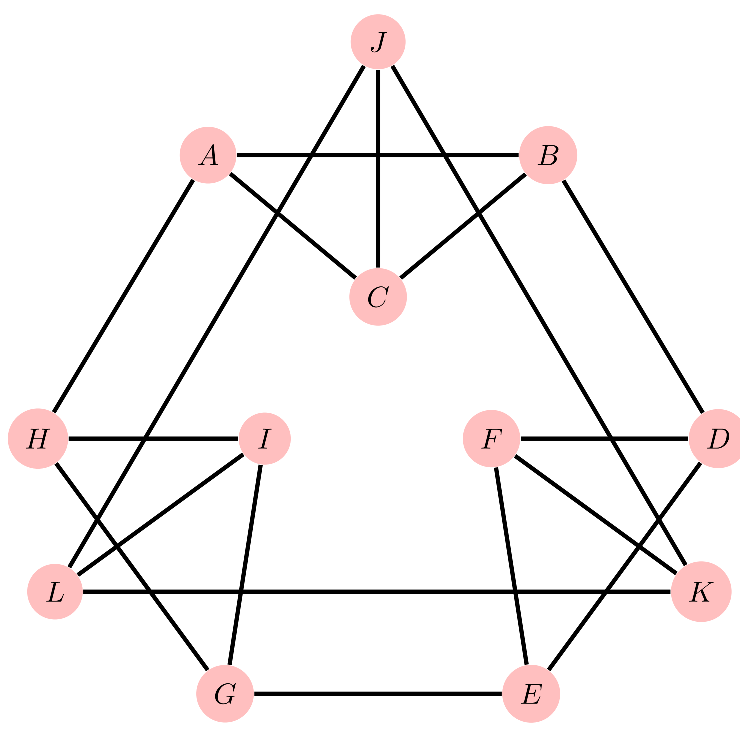
(e) Is the FourTriangles graph a bipartite graph?

Answer: No. Because bipartite graphs do not have triangles and the FourTriangles graph has four triangles.

The claim about cycles of size 3 can be proven as follows for a triangle X, Y, Z. If X is on one side of the bipartite graph, then its neighbors Y and Z must be on the other side of the bipartite graph. This is a contradiction, because there is an edge between Y and Z and therefore they cannot be on the same side of the bipartite graph.

The above proof can be generalized to any cycle of an odd size proving that a cycle of an odd size is not a bipartite graph.

(f) The following assignment of the 12 labels $A, B, C, D, E, F, G, H, I, J, K, L$ to the 12 vertices in the drawing below demonstrates that this drawing is isomorphic to the original drawing of the FourTriangles graph. That is, for any pair of vertices $u, v \in \{A, B, C, D, E, F, G, H, I, J, K, L\}$, the edge (u, v) exists in the original drawing if and only if it exists in the drawing below.

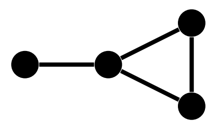


14. A **balloon-graph** is a graph that is composed of a cycle that is connected to a path.

- All the edges of the graph are either part of the cycle or part of the path.
- Exactly one vertex is shared by the cycle and the path.
- The cycle has at least 3 vertices and therefore at least 2 vertices belong only to the cycle.
- The path has at least 2 vertices and therefore at least 1 vertex belongs only to the path.

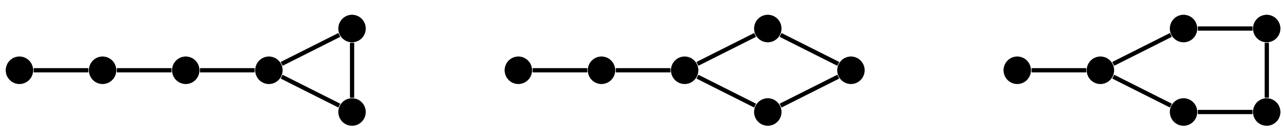
Observation: A balloon-graph must contain at least 4 vertices: the shared vertex, at least 2 additional vertices in the cycle, and at least 1 additional vertex in the path.

(a)

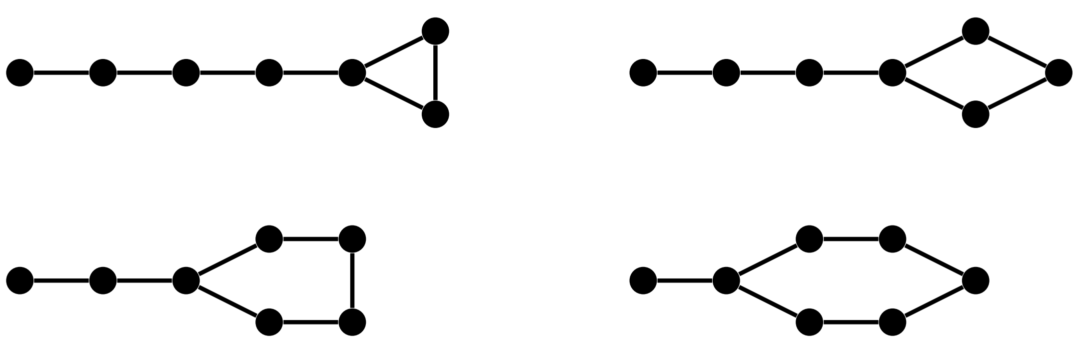


The only possible balloon-graph with 4 vertices.

The two non-isomorphic balloon-graphs with 5 vertices.



The three non-isomorphic balloon-graph with 6 vertices.



The four non-isomorphic balloon-graph with 7 vertices.

(b) For $n \geq 4$, how many non-isomorphic balloon-graphs with n vertices exist?

Answer: The size of the cycle, denoted by c, determines the length of the path, denoted by p. Since the cycle and the path share exactly one vertex, it follows that $n = c + p - 1$. A cycle must contain at least three vertices and therefore $c \geq 3$ while the path must contain at least two vertices and therefore $p \geq 2$ which implies that $c \leq n - 1$. As a result, there are exactly $n - 3$ balloon-graphs each determined by $c = 3, 4, \ldots, n - 1$.

(c) For $n \geq 4$, all balloon-graphs with n vertices have the same number of edges. What is this number?

Answer: Denote by c the size of the cycle and by p the length of the path. In the previous part, it was shown that $n = c + p - 1$. A cycle with c vertices has c edges and a path of length p has $p - 1$ edges. The edges of the cycle are disjoint to the edges of the path. Hence, the number of edges m of a balloon-graph with n vertices is $c + p - 1$. That is, $m = n$ in a balloon-graph.

(d) For $n \geq 4$, the degree sequence of all possible balloon-graphs is the same. Let G be a balloon-graph with n vertices. How many vertices in G have degree 0? How many vertices in G have degree 1? How many vertices in G have degree 2? How many vertices in G have degree 3? How many vertices in G have degree greater than 3?

Answer: Denote by c the size of the cycle and by p the length of the path. Let v be the vertex shared by the cycle and the path.

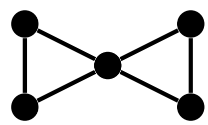
i. The degree of v is 3.
ii. The degrees of any of the other $c - 1$ vertices in the cycle are 2
iii. The degree of the vertex at the other end of the path is 1.
iv. the degrees of the $p - 2$ inner vertices in the path are 2.

It follows that a balloon graph with n vertices has zero vertices with degree 0, one vertex with degree 1, $n - 2 = (c - 1) + (p - 2)$ vertices with degree 2, and one vertex with degree 3. None of the vertices have degree greater than 3.

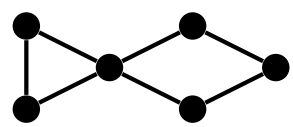
15. An **eight-graph** is a graph that is composed of two cycles that share exactly 1 vertex.
    - All the edges of the graph are part of one of the two cycles.
    - Exactly one vertex is shared by both cycles.
    - The rest of the vertices belong only to one of the cycles.
    - A cycle must contain at least 3 vertices including the one that connects both cycles.

Observation: An eight-graph must contain at least 5 vertices: the shared vertex and at least 2 additional vertices in each cycle.

(a)



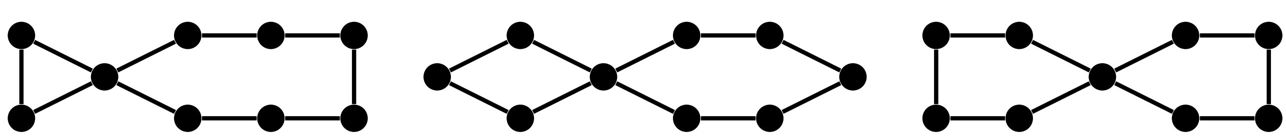
The only possible eight-graph with 5 vertices.



The only possible eight-graph with 6 vertices.

The two non-isomorphic eight-graph with 7 vertices.

The two non-isomorphic eight-graph with 8 vertices.



The three non-isomorphic eight-graph with 9 vertices.

(b) For $n \geq 5$, how many non-isomorphic eight-graphs with n vertices exist?

Answer: The size of one of the cycles, denoted by c_1, determines the size of the other cycle, denoted by c_2. Since the two cycles share exactly one vertex, it follows that $n = c_1 + c_2 - 1$. A cycle must contain at least three vertices and therefore $c_1 \geq 3$ and $c_2 \geq 3$ which implies that $c_1 \leq n - 2$. As a result, there are exactly $n - 4$ possible values for c_1 from the range $3, 4, \ldots, n - 2$. However, the eight-graph in which the size of one cycle is c and the size of the other cycle is $n + 1 - c$ is counted twice once when $c_1 = c$ and once when $c_1 = n + 1 - c$.

It follows that for an even n the $n/2 - 2$ non-isomorphic sizes for c_1 are $3, 4, \ldots, n/2$ and for an odd n the $(n + 1)/2 - 2$ non-isomorphic sizes for c_1 are $3, 4, \ldots, (n + 1)/2$. This implies that the number of non-isomorphic eight-graphs with n vertices is

$$\left\lfloor \frac{n+1}{2} \right\rfloor - 2 = \left\lfloor \frac{n-3}{2} \right\rfloor$$

Examples: For $n = 5$ there are $\lfloor \frac{5-3}{2} \rfloor = 1$ non-isomorphic eight-graphs, for $n = 6$ there are $\lfloor \frac{6-3}{2} \rfloor = 1$ non-isomorphic eight-graphs, for $n = 7$ there are $\lfloor \frac{7-3}{2} \rfloor = 2$ non-isomorphic eight-graphs, for $n = 8$ there are $\lfloor \frac{8-3}{2} \rfloor = 2$, non-isomorphic eight-graphs and for $n = 9$ there are $\lfloor \frac{9-3}{2} \rfloor = 3$ non-isomorphic eight-graphs.

(c) For $n \geq 5$, all eight-graphs with n vertices have the same number of edges. What is this number?

Answer: Denote by c_1 and c_2 the sizes of the two cycles. In the previous part, it was shown that $n = c_1 + c_2 - 1$. A cycle with c vertices has c edges. The edges of both cycles are disjoint. Hence, the number of edges m of an eight-graph with n vertices is $c_1 + c_2$. That is, $m = n + 1$ in an eight-graph.

(d) For $n \geq 5$, the degree sequence of all possible eight-graphs is the same. Let G be an eight-graph with n vertices. How many vertices in G have degree 0? How many vertices in G have degree 1? How many vertices in G have degree 2? How many vertices in G have degree 3? How many vertices in G have degree 4? How many vertices in G have degree greater than 4?

Answer: Let v be the vertex shared by the two cycles. It follows that the degree of v is 4 while the degrees of any of the other $n - 1$ vertices in the cycles are 2.

Therefore, an eight-graph with n vertices has zero vertices with degree 0 or 1, $n - 1$ vertices with degree 2, zero vertices with degree 3, one vertex with degree 4. None of the vertices have degree greater than 4.

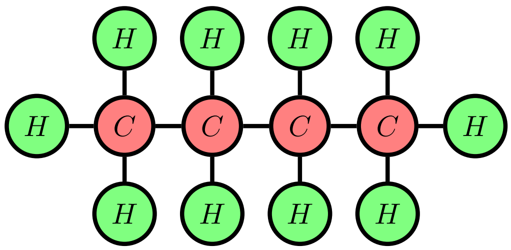
16. There are six possible molecules of the type C_4H_y for some positive integer y in which a connection between two C-atoms can have at most one edge, the degree of each one of the C-atoms is 4, and the degree of each one of the H-atoms is 1.

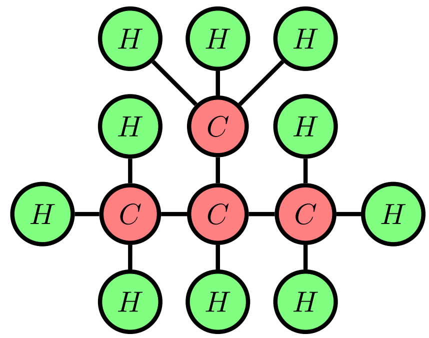
Observation 1: Because the degree of the H-atoms is 1, the molecule is connected only if the sub-graph containing the four C-atoms is connected. Therefore, the number of inter-connections among the C-atoms must be at least 3. This number is at most 6 because there are four C-atoms.

Observation 2: Assume that the number of connections (edges) among the C-atoms is x. Each one of the H-atoms contributes one edge to the total number of edges which is therefore, $m = x + y$. By the hand-shaking lemma the sum of the degrees which is $4 \cdot 4 + y \cdot 1 = 16 + y$ is equal to $2m$. That is,

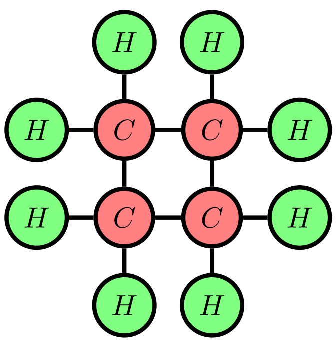
$$16 + y = 2(x + y) \quad \Longrightarrow \quad y = 16 - 2x$$

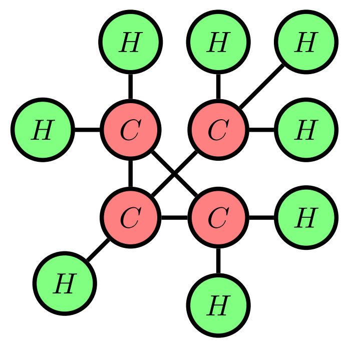
The two C_4H_{10} molecules: By Observation 2, there are ten H-atoms ($y = 10$) if there are three connections among the C-atoms ($x = 3$). In the left molecule below the C-sub-graph is a path and in the right molecule below the C-sub-graph is a star.



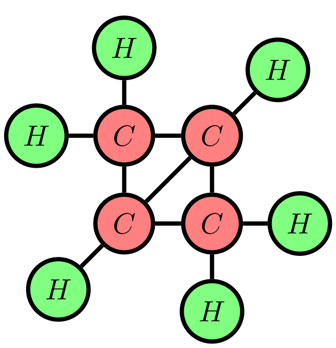


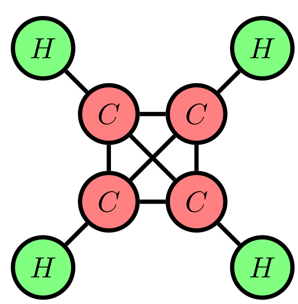
The two C_4H_8 molecules: By Observation 2, there are eight H-atoms ($y = 8$) if there are four connections among the C-atoms ($x = 4$). In the left molecule below the C-sub-graph is a cycle and in the right molecule below the C-sub-graph is a triangle plus an edge.





The C_4H_6 and C_4H_4 molecules: By Observation 2, there are six H-atoms ($y = 6$) if there are five connections among the C-atoms ($x = 5$) and four H-atoms ($y = 4$) if there are six connections among the C-atoms ($x = 6$). In the left molecule below the C-sub-graph is the complete graph minus one edge and in the right molecule below the C-sub-graph is the complete graph.



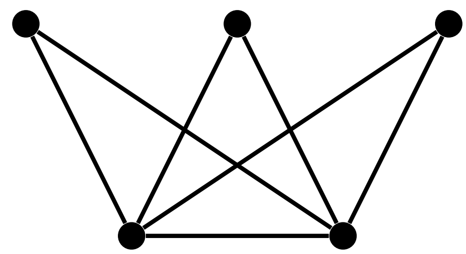


17. Let G be a simple (no parallel edges and no self loops) undirected graph on the $n \geq 1$ vertices $\{v_1, v_2, \ldots, v_n\}$.

If vertex v_i has d_i neighbors (equivalently, the degree of v_i is d_i), then the **degree sequence** of G is $D_G = (d_1, d_2, \ldots, d_n)$. If a sequence D has a graph G for which D is G's degree sequence then D is a **graphic** sequence and G is a **realization** of D.

One of the following three degree sequences is not a graphic sequence, one of them is a graphic sequence with exactly one realization, and one of them is a graphic sequence with two non-isomorphic realizations. Which is which?

(a) The sequence $(4, 4, 2, 2, 2)$ has only one realization. This is because each one of the two vertices with degree 4 must be connected to all the other four vertices. As a result of these edges, the other three vertices already have degree 2. There is no need for additional edges to realize the sequence and therefore the graph below is the only realization graph for the sequence $(4, 4, 2, 2, 2)$.

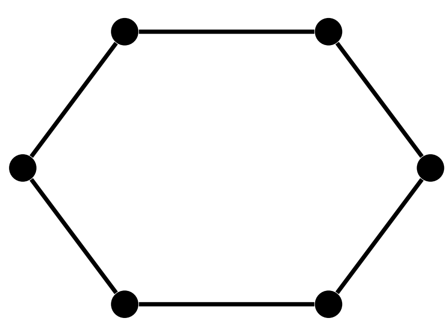


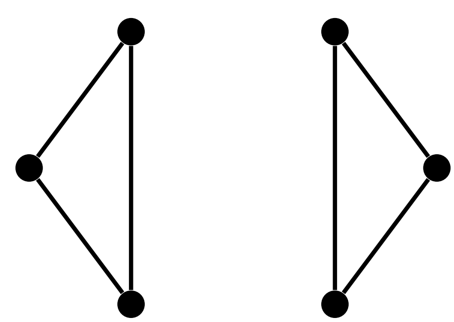
(b) The sequence $(4, 4, 1, 1, 1, 1)$ is not graphic.

Proof 1: Since there are six vertices, the two vertices with degree 4 must share at least two neighbors among the remaining four vertices because each requires at least three neighbors even if both are connected by an edge. This implies that two additional vertices must have a degree at least 2. This is a contradiction because all the other degrees are 1.

Proof 2: Run the algorithm that constructs a realization for a graphic sequence or aborts if such a realization does not exist. After the first recursive call, the sorted sequence becomes $(3, 1, 0, 0, 0)$. Now since the largest degree is 3 and there is only one additional non-zero degree, the algorithm aborts without a realization.

(c) The sequence $(2, 2, 2, 2, 2, 2)$ has two non-isomorphic realization. In one of them the graph contains a cycle of length 6 while the other graph contains two cycles.

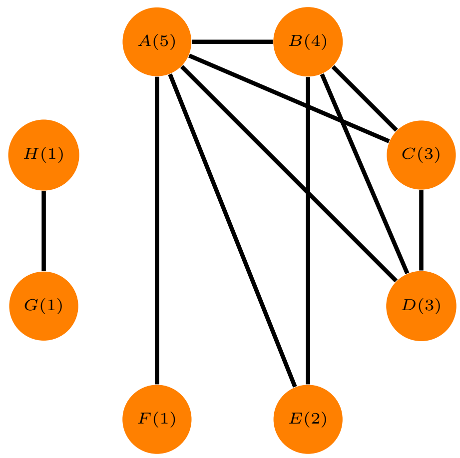




18. The sequence $S = (5, 4, 3, 3, 2, 1, 1, 1)$ is graphic. Associate the eight vertices A, B, C, D, E, F, G, H with the degrees by their non-increasing order.

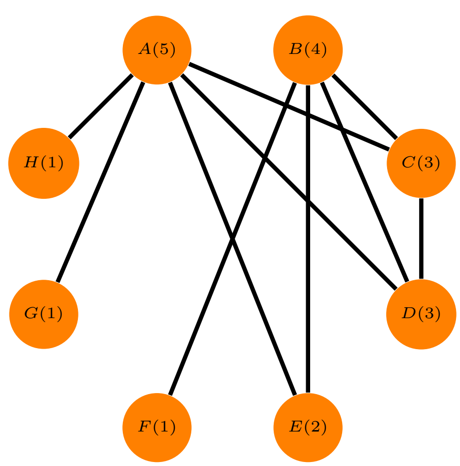
(a) Construct the realization graph of the sequence S by running the algorithm in which the pivot is always the vertex with the current highest degree. When there are more than one candidate for the pivot or when there are several choices for selecting the neighbors for the pivot, give preference to the vertices which appear first in the alphabetical order from A to H.

- Round 1: input sequence is $(5, 4, 3, 3, 2, 1, 1, 1)$ and the pivot A is connected to B, C, D, E, and F.
- Round 2: input sequence is $(0, 3, 2, 2, 1, 0, 1, 1)$ and the pivot is B that is connected to C, D, and E.
- Round 3: input sequence is $(0, 0, 1, 1, 0, 0, 1, 1)$ and the pivot is C that is connected to D.
- Round 4: input sequence is $(0, 0, 0, 0, 0, 0, 1, 1)$ and the pivot is G that is connected to H.
- After round 4 the sequence is $(0, 0, 0, 0, 0, 0, 0, 0)$ and the algorithm terminates.



(b) Construct the realization graph of the sequence S by running the algorithm in which the pivot is always the vertex with the current lowest degree. When there is more than one candidate for the pivot, give preference to the vertex which appear last in the alphabetical order from A to H. However, when there are several choices for selecting the neighbors for the pivot, give preference to the vertices which appear first in the alphabetical order from A to H.

- Round 1: input sequence is $(5, 4, 3, 3, 2, 1, 1, 1)$ and the pivot is H that is connected to A.
- Round 2: input sequence is $(4, 4, 3, 3, 2, 1, 1, 0)$ and the pivot is G that is connected to A.
- Round 3: input sequence is $(3, 4, 3, 3, 2, 1, 0, 0)$ and the pivot is F that is connected to B.
- Round 4: input sequence is $(3, 3, 3, 3, 2, 0, 0, 0)$ and the pivot is E that is connected to A and B.
- Round 5: input sequence is $(2, 2, 3, 3, 0, 0, 0, 0)$ and the pivot is B that is connected to C and D.
- Round 6: input sequence is $(2, 0, 2, 2, 0, 0, 0, 0)$ and the pivot is D that is connected to A and C.
- Round 7: input sequence is $(1, 0, 1, 0, 0, 0, 0, 0)$ and the pivot is C that is connected to A.
- After round 7 the sequence is $(0, 0, 0, 0, 0, 0, 0, 0)$ and the algorithm terminates.

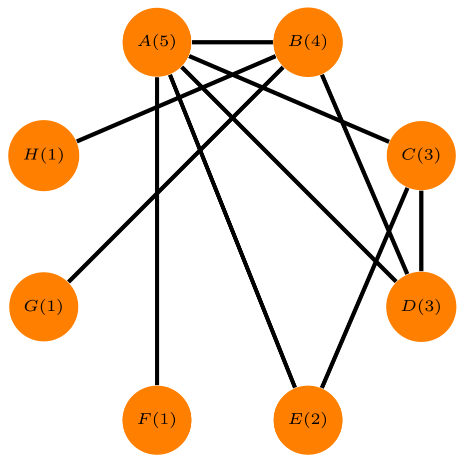


(c) The two realizations G_a in part (a) and G_b in part (b) are not isomorphic. The following are four differences.

- In G_a the vertex with degree 4 is connected to the vertex with degree 4. This is not the case in G_b.
- In G_a two of the vertices with degree 1 are connected to each other. This is not the case in G_b.
- G_a has two connected components while G_b is a connected graph.
- In G_a there is a complete sub-graph with four vertices. This is not the case in G_b.

(d) Find another sequence of pivots (recall that in any recursive round, any vertex may be the pivot as long as it is connected to the current highest degree vertices) that generates a third non-isomorphic realization graph of the sequence S.

- Round 1: input sequence is $(5, 4, 3, 3, 2, 1, 1, 1)$ and the pivot A is connected to B, C, D, E, and F.
- Round 2: input sequence is $(0, 3, 2, 2, 1, 0, 1, 1)$ and the pivot is H that is connected to B.
- Round 3: input sequence is $(0, 2, 2, 2, 1, 0, 1, 0)$ and the pivot is G that is connected to B.
- Round 4: input sequence is $(0, 1, 2, 2, 1, 0, 0, 0)$ and the pivot is C that is connected to D and E.
- Round 5: input sequence is $(0, 1, 0, 1, 0, 0, 0, 0)$ and the pivot is D that is connected to B.
- After round 5 the sequence is $(0, 0, 0, 0, 0, 0, 0, 0)$ and the algorithm terminates.



(e) The realization G_d in part (d) is isomorphic neither to the realization G_a in part (a) nor to the realization G_b in part (b). This is because in G_d the vertex with degree 4 is connected to two vertices with degree 1 while in G_a none of the neighbors of the vertex with degree 4 has degree 1 and in G_b the vertex with degree 4 is connected to only one vertex with degree 1.

19. A *coloring* of a graph is an assignment of "colors" to the vertices such that the colors assigned to the two vertices of any edge are different.

Notations: Let the vertices of the graph be $v_0, v_1, \ldots, v_{n-1}$. Let the colors be the positive integers $1, 2, \ldots$. Let the color assigned to v_i be $c(v_i)$ or c_i. For a given graph G, let $\chi(G)$ be the minimum number of colors that are required to color all the vertices of G.

(a) How many colors are needed to color the null graph (N_n) with $n \geq 1$ vertices?

Answer: $\chi(N_n) = 1$. One color that is assigned to all the n vertices is enough. There are no conflicts because there are no edges. Therefore, $c_i = 1$ for all $1 \leq i \leq n$ is an optimal coloring of N_n.

(b) How many colors are needed to color the complete graph (K_n) with n vertices?

Answer: $\chi(K_n) = n$. Each vertex must have a different color since all possible edges exist. Therefore, $c_i = i$ for all $1 \leq i \leq n$ is an optimal coloring of K_n.

(c) How many colors are needed to color a tree (T_n) with n vertices?

Answer: Trivially, $\chi(T_1) = 1$ for a tree with one vertex. For $n \geq 2$, $\chi(T_n) = 2$. Color an arbitrary vertex with the color 1. Then color its neighbors with the color 2. Repeat the following until all the vertices of the graph are colored: color with 2 all the neighbors of a newly colored by 1 vertex and color with 1 all the neighbors of a newly colored by 2 vertex.

Since there are no cycles in a tree, it follows that each edge is reached once by the above process. The first vertex in the edge is colored by 1 or 2 while the second vertex is colored by 2 or 1 respectively.

A graph with at least one edge requires at least two colors. Therefore, the above coloring of the tree with two colors is optimal.

(d) How many colors are needed to color a bipartite (B_n) graph?

Answer: $\chi(B_n) = 2$ for a non-null bipartite graph. By definition, the vertices of a bipartite graph are composed of two sets X and Y such that all the edges in the graph are between a vertex from X and a vertex from Y. As a result, coloring all the vertices of X with 1 and all the vertices of Y with 2 is a coloring.

If the bipartite graph has at least one edge, this edge already requires two colors. Therefore, the above coloring is optimal for all non-null bipartite graphs.

(e) How many colors are needed to color the cycle graph (C_n) with n vertices?

Answer: $\chi(C_n) = 2$ for an even $n \geq 2$ and $\chi(C_n) = 3$ for an odd $n \geq 3$.

Let the edges of the cycle be (v_i, v_{i+1}) for $0 \leq i \leq n-1$ where $(n-1)+1 = 0$. Consider the path graph $P_n = (v_0, v_1, \ldots, v_{n-1})$ (the cycle C_n without the edge (v_{n-1}, v_0)). P_n is a tree and therefore can be colored with the colors 1 and 2 as described in part (c).

Observe that if the tree-coloring process starts with v_0, then $c_0 = 1$ and $c_{n-1} = 2$ if n is even while $c_{n-1} = 1$ if n is odd.

Assume n is even. Then by adding back the edge (v_{n-1}, v_0) to the path P_n and maintaining the colors, a coloring of the cycle C_n with two colors is obtained since $c_0 \neq c_{n-1}$.

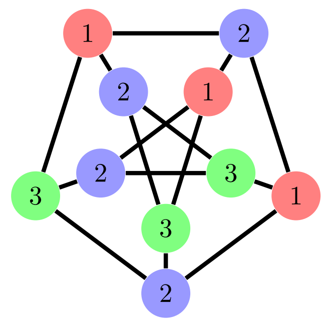
Assume n is odd. By changing the color of c_{n-1} to 3 and maintaining the colors of the other vertices from the path coloring, a coloring of the cycle C_n with three colors is obtained since $c_0 \neq c_{n-1}$.

Finally, assume to the contrary that C_n can be colored with two colors for an odd $n \geq 3$. It follows that the colors of all the even indexed vertices $v_0, v_2, \ldots, v_{n-1}$ must be the same. This is a contradiction since c_0 must be different than c_{n-1}.

(f) Color the Petersen graph with as few colors as possible.

Answer: See below a coloring of the Petersen graph with three colors.

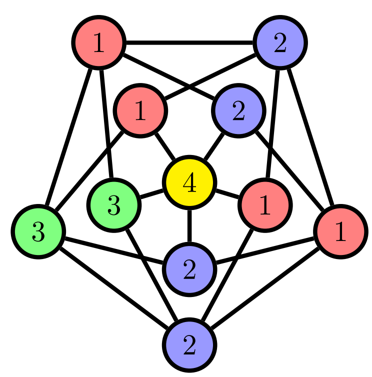
In part (e), it was proven that an odd-length cycle cannot be colored with two colors. The Petersen graph contains two cycles of length 5: (A, B, C, D, E) and (F, H, J, G, I). Each of these cycles requires three colors. Therefore, even without the five cycle-connecting edges (A, F), (B, G), (C, H), (D, I), and (E, J), a coloring of the Petersen graph requires at least three colors.



(g) Color the Grötzsch graph with as few as possible colors.

Answer: See below a coloring of the Grötzsch graph with four colors.

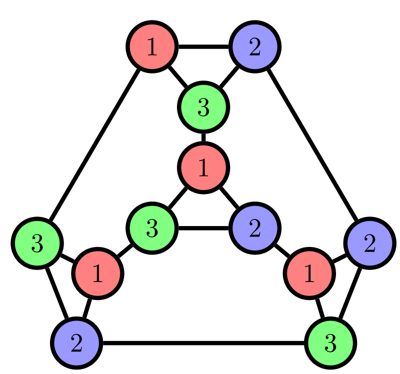
Assume to the contrary that it is possible to color the Grötzsch graph with three colors. Without loss of generality assume that K is colored with the color 3. Therefore, F, G, H, I, J must be colored with the colors 1 and 2. Change the color of A to the color of F. Since the neighbors B and E of A are also the neighbors of F, it follows that the new color of A is different then the colors of B and E. Similarly, change the color of G to the color of B, the color of H to the color of C, the color of I to the color of D, and the color of J to the color of E. As a result the vertices in the cycle (A, B, C, D, E, A) are colored only with the colors 1 and 2. A contradiction to the fact that a cycle of size 5 cannot be colored with two colors (part (e)).



(h) Color the FourTriangle graph with as few as possible colors.

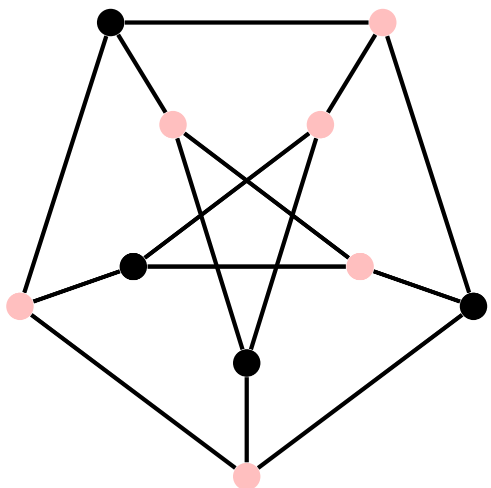
Answer: See below a coloring of the FourTriangles graph with three colors.

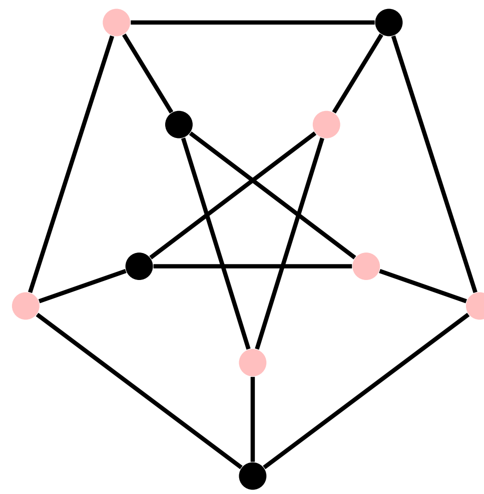
The FourTriangle graph contains four cycles of size 3 (triangles). Each triangle by itself requires three colors (see part (e)). Therefore, it is impossible to color the whole graph with only two colors.



20. Let G be a simple undirected graph with $n \geq 1$ vertices. An **independent set** in G is a set of vertices I that have no edges among themselves. That is, for any pair of vertices $u, v \in I$, the graph G does not include the edge (u, v).

Answer: 4. See below two examples. The black vertices are the independent sets.





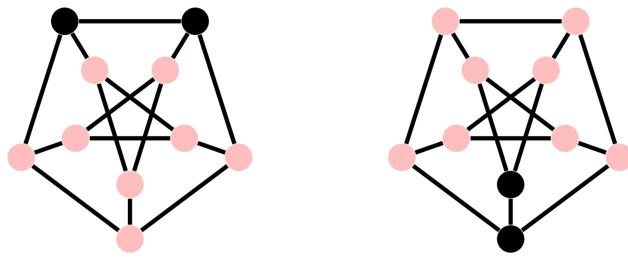
Proposition: A larger set does not exist.

Proof: The largest independent set in the cycle of size 5, denoted by C_5, is 2. To see this, let v be one of the vertices in an independent set of C_5. Then v's two neighbors cannot be in this independent set and since the remaining two vertices are neighbors only one of them can be in this independent set. The Petersen graph has an outer cycle $C_5 = (A, B, C, D, E)$ and an inner cycle $C_5 = (F, H, J, G, I)$. Each cycle may "contribute" at most two vertices to the independent set even without the additional restrictions imposed by the edges that connect both cycles. Therefore, the size of the maximum independent set in the Petersen graph is at most 4.

Remark: Up to "symmetry" there is only one independent set of size 4 in the unlabeled Petersen graph. Such an independent set must contain two non-adjacent vertices in the outer cycle and two non-adjacent vertices in the inner cycle. Moreover, a pair of non-adjacent vertices in any of the cycle can be combined with only one pair of non-adjacent vertices in the other cycle. This implies that there are exactly five different independent sets of size 4 in the labeled Petersen graph: (A, C, I, J) (the left example above), (A, D, G, H), (B, D, F, J) (the right example above), (B, E, H, I), and (C, E, F, G).

21. Let G be a simple undirected graph with $n \geq 1$ vertices. A **clique** in G is a set of vertices C that contains all the possible edges among themselves. That is, for any pair of vertices $u, v \in C$, the graph G includes the edge (u, v).

Answer: 2. See below two examples. The black vertices are the cliques.

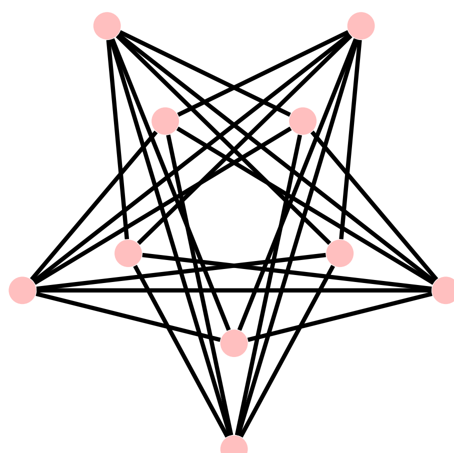


Proposition: A larger set does not exist.

Proof: By inspecting all possible triplets of vertices, it can be shown that the Petersen graph does not have a triangle graph. Therefore, any edge is the largest clique in the Petersen graph.

Remark: Up to "symmetry" there is only one clique of size 2 in the unlabeled Petersen graph. Such a clique is an edge. Since the Petersen graph has ten edges, this implies that there are exactly ten different cliques of size 2 in the labeled Petersen graph.

Definition: Let G be a graph. The complement graph of G, denoted by $\tilde{G}$, is a graph with the same set of vertices whose edges are exactly all the edges that are not in G. Note, that by definition G is the complement graph of $\tilde{G}$. See below the complement of the Petersen graph.



Cliques vs. independent sets: C is a clique in a graph G iff C is an independent set in $\tilde{G}$.

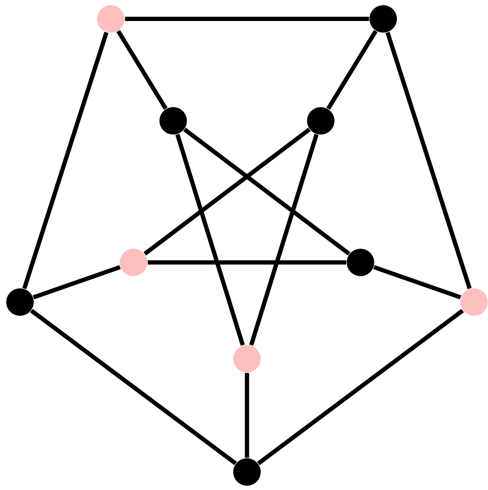
Proof:

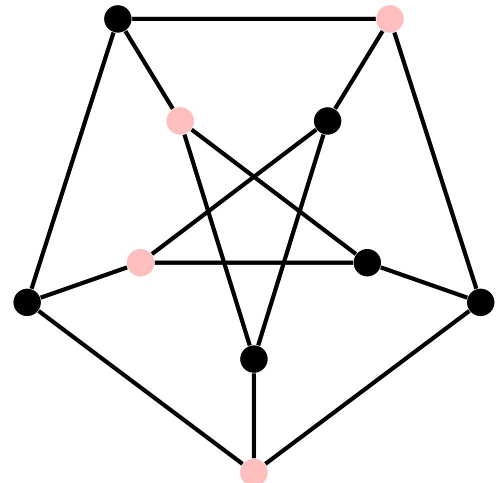
- For the if direction of the iff statement, assume that C is a clique in G. By definition, C contains all possible edges between pair of vertices in C. Therefore, C does not contain a pair of vertices that has an edge in $\tilde{G}$. Hence C is an independent set in $\tilde{G}$.
- For the only if direction of the iff statement, assume that C is an independent set in $\tilde{G}$. By definition, C doesn't contain a pair of vertices that are connected by an edge in $\tilde{G}$. Therefore, C in G contains all possible edges between pair of vertices from C. Hence, C is a clique in G.

It follows that the size of the largest independent set in a graph is the same as the size of the largest clique in its complement graph. One can verify that the size of the largest clique in the complement of the Petersen graph is 4 the same as the size of the largest independent set in the Petersen graph and that the size of the largest independent set in the complement of the Petersen graph is 2 the same as the size of the largest clique in the Petersen graph.

22. Let G be a simple undirected graph with $n \geq 1$ vertices. A **vertex cover** in G is a set of vertices VC for which each edge has at least one of its vertices in the set. That is, for any edge (u, v) of G either $u \in VC$ or $v \in VC$.

Answer: 6. See below two examples. The black vertices are the vertex covers.





Proposition: A larger set does not exist.

Proof: The smallest vertex cover in the cycle of size 5, denoted by C_5, is 3 since each vertex can cover at most two edges and the cycle has five edges. The Petersen graph has an outer cycle $C_5 = (A, B, C, D, E)$ and an inner cycle $C_5 = (F, H, J, G, I)$. Each cycle must "contribute" at least three vertices to the vertex cover even without the additional restrictions imposed by the edges that connect both cycles. Therefore, the size of the minimum vertex cover in the Petersen graph is at least 6.

Remark: Up to "symmetry" there is only one vertex cover of size 6 in the unlabeled Petersen graph. Such a vertex cover must contain three vertices in the outer cycle such that one is not adjacent to the other two which are neighbors and three vertices in the inner cycle such that one is not adjacent to the other two which are neighbors. Moreover, in order to cover the five edges that connect the two cycles, any such triplet of vertices in any of the cycle can be combined with only one such triplet of vertices in the other cycle. This implies that there are exactly five different independent sets of size 6 in the labeled Petersen graph: (B, D, E, F, G, H) (the left example above), (A, C, D, F, G, J), (A, C, E, G, H, I) (the right example above), (A, B, D, H, I, J), and (B, C, E, F, I, J).

Vertex covers vs. independent sets: Let G be a graph and let V be the set of all the vertices of G. Then I is an independent set in G iff $VC = V \setminus I$ is a vertex cover in G.

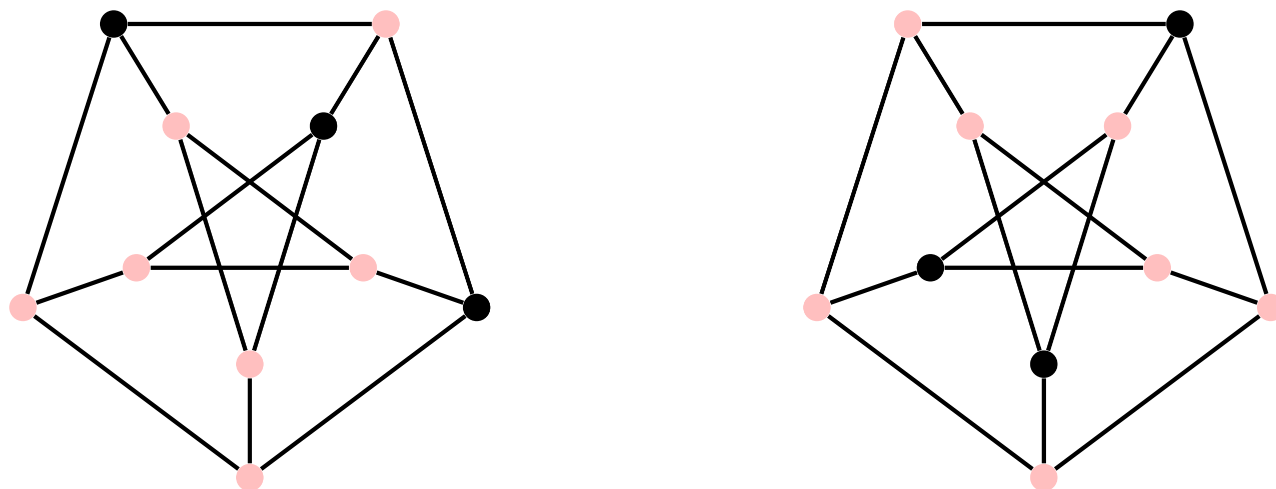
Proof:

- For the if direction of the iff statement, assume that I is an independent set in G. By definition, I cannot contain two vertices connected by an edge. As a result, each edge has at least one vertex that does not belong to I and therefore belongs to VC which implies that VC is a vertex cover in G.
- For the only if direction of the iff statement, assume that VC is a vertex cover in G. By definition, VC must contain a representative vertex from each edge. Hence, I cannot contain the two endpoints of any edge which implies that I is an independent set.

It follows that the complement of an independent set with the largest size in the graph is a vertex cover with the smallest size in the graph. As a result, for any graph G, the size of G's largest independent set plus the size of G's smallest vertex cover is the number of vertices in G. Indeed, the Petersen graph has 10 vertices, the size of the largest independent set is 4, and the size of the smallest vertex cover is 6.

23. Let G be a simple undirected graph with $n \geq 1$ vertices. A **dominating set** in G is a set of vertices D for which all other vertices have at least one neighbor in D. That is, for any vertex v of G either $v \in D$ or there exists a neighbor U of v such that $u \in D$.

Answer: 3. See below two examples. The black vertices are the dominating sets.



Proposition: A larger set does not exist.

Proof: Each vertex can dominate at most three other vertices since the degree of each vertex is 3. Therefore, a dominating set with two vertices can dominate at most eight vertices including themselves. As a result, a dominating set in the Petersen graph must contain at least 3 vertices.

Remark: Up to "symmetry" there is only one dominating of size 3 in the unlabeled Petersen graph. Such a dominating set must contain two non-adjacent vertices either in the outer cycle or in the inner cycle and one vertex in the other cycle. Moreover, a pair of non-adjacent vertices in any of the cycles can be combined with only one vertex in the other cycle. This implies that there are exactly $10 = 5 \cdot 2$ different dominating sets of size 3 in the labeled Petersen graph: (A, C, G) (the left example above), (B, D, H), (C, E, I), (A, D, J), (B, F, E), (A, H, I), $(B, I, J,)$ (the right example above), (C, F, J), (D, F, G), and (E, G, H).

The maximum distance between any two vertices in the Petersen graph is 2. That is, starting with any vertex, it is possible to reach any other vertex via a path that is composed of two adjacent edges. As a result, any set that includes all the three neighbors of a particular vertex is a dominating set. Indeed, each one of the above ten different dominating sets in the labeled Petersen graph is a set that includes exactly the three neighbors of one of the vertices in the graph.

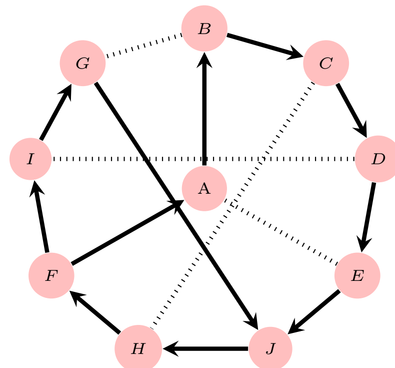
Dominating sets vs. vertex covers: Every vertex cover without singleton vertices (a vertex without edges) is also a dominating set since if a vertex is not in the vertex cover it must be dominated by one of its neighbors all of which must be in the vertex cover. The other direction is false. For example, any vertex cover of the complete graph must contain all the vertices except once. Because if it does not contain two vertices, the edge between them would not be covered. As a result, the size of the smallest vertex cover for the complete graph with n vertices is $n - 1$. However, any vertex of the complete graph dominates the rest of the vertices. As a result, the size of the smallest dominating set for the complete graph with n vertices is 1.

24. A **path** $\mathcal{P} = \langle v_0, v_1, \ldots, v_k \rangle$ of length k (number of edges) is an ordered list of $k+1$ vertices such that the edge (v_i, v_{i+1}) exists for $0 \le i \le k-1$ and all the edges are different. A **cycle** $\mathcal{C} = \langle v_0, v_1, \ldots, v_{k-1}, v_0 \rangle$ of length k is a path that starts and ends with the same vertex.

Remark: The answers to the following two questions are shown on another drawing of the Petersen graph which is isomorphic to the traditional drawing.

(a) Find one of the longest paths (does not have to be simple) in the Petersen graph.

Answer: $(F - A - B - C - D - E - J - H - F - I - G - J)$ is a path that contains 11 edges. The four edges that are not in the path are (A, E), (C, H), (B, G), and (D, I).

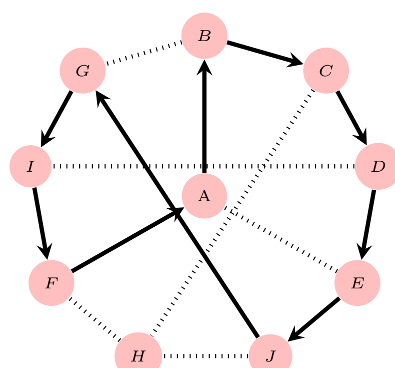


Proposition: The length of the longest path in the Petersen graphs is 11.

Proof: In any path, the path-degree of each vertex is even except the end vertices whose path-degree is odd. Therefore, in any path in the Petersen graph, for which the degree of each vertex is 3, the path-degree of two vertices is 1 or 3 while the path-degree of the rest of the vertices is 2. Observe that the number of edges in any path is the sum of the path-degrees of the vertices divided by 2 since each edge is counted twice. Therefore, the longest possible path in the Petersen graph would contain eight vertices with one appearance in the path and two vertices (the end vertices) with two appearances in the path. Such a path would have $(2 \cdot 3 + 8 \cdot 2)/2 = 11$ edges.

(b) Find one of the longest cycles (does not have to be simple) in the Petersen graph.

Answer: $(A - B - C - D - E - J - G - I - F - A)$ is a simple cycle that contains 9 edges and 9 vertices. The only vertex that is not in the cycle is H.



Proposition: The length of the longest cycle in the Petersen graphs is 9.

Proof: In any cycle the degree of each vertex is even. Therefore, in any cycle in the Petersen graph, the cycle-degree of each vertex is exactly 2. As a result, any cycle in the Petersen graph is also a simple cycle in which each vertex appears at most once in the cycle. The Petersen graph does not contain a simple cycle of length 10 (a Hamiltonian cycle) and therefore the above cycle is one of the longest cycle in the Petersen graph.

See `https://www.youtube.com/watch?v=AVe-OA-fcVO` for an elegant proof that the Petersen graph does not have a Hamiltonian cycle.

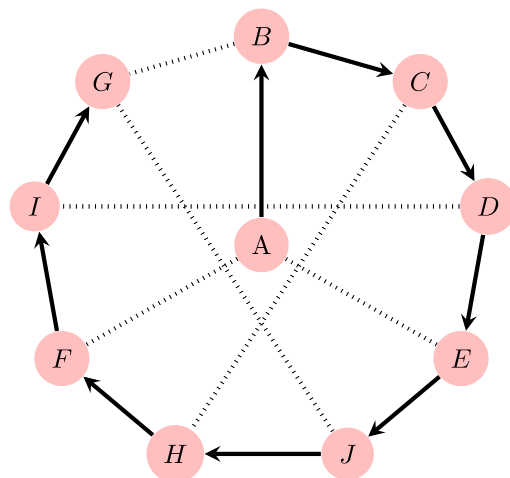
25. A **path** $\mathcal{P} = \langle v_0, v_1, \ldots, v_k \rangle$ of length k (number of edges) is an ordered list of $k+1$ vertices such that the edge (v_i, v_{i+1}) exists for $0 \leq i \leq k-1$ and all the edges are different. A **cycle** $\mathcal{C} = \langle v_0, v_1, \ldots, v_{k-1}, v_0 \rangle$ of length k is a path that starts and ends with the same vertex. In a **simple path** all the vertices are different. In a **simple cycle** all the vertices except $v_0 = v_k$ are different.

A **Hamiltonian path** is a simple path that contains all the vertices in the graph. A **Hamiltonian cycle** is a simple cycle that contains all the vertices in the graph.

Remark: The answers to the following two questions are shown on another drawing of the Petersen graph which is isomorphic to the traditional drawing.

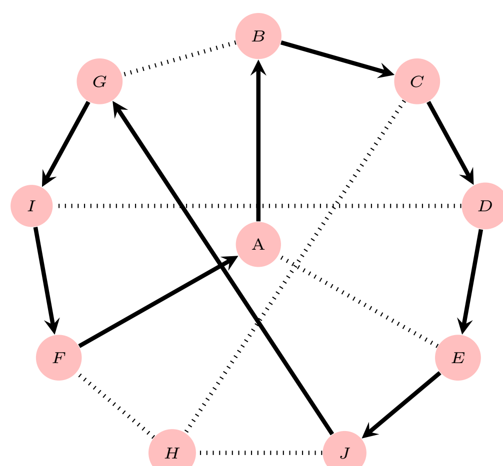
(a) Find one of the Hamiltonian paths in the Petersen graph.

Answer: $(A - B - C - D - E - J - H - F - I - G)$ is one of the Hamiltonian paths of the Petersen graph. This path starts at A and ends at H.



(b) Find one of the longest simple cycles in the Petersen graph.

Answer: $(A - B - C - D - E - J - G - I - F - A)$ is a simple cycle that contains 9 edges and 9 vertices. The only vertex that is not in the cycle is H.



Proposition: The length of the longest simple cycle in the Petersen graphs is 9.

Proof: The Petersen graph does not have a Hamiltonian cycle (a cycle of length 10). Therefore, a cycle of length 9 shown above is the longest simple cycle in the Petersen graph.

See `https://www.youtube.com/watch?v=AVe-0A-fcV0` for an elegant proof that the Petersen graph does not have a Hamiltonian cycle.

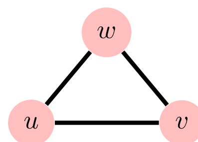
26. Trees and bipartite graphs.

Notations: Define the following sets of graphs:

- $\mathcal{G}$: the set of all graphs.
- $\mathcal{B}$: the set of all bipartite graphs.
- $\mathcal{T}$: the set of all tree graphs.

Observation: $\mathcal{T} \subset \mathcal{G}$ and $\mathcal{B} \subset \mathcal{G}$: There are graphs that are not trees and there are graphs that are not bipartite graphs.

Proof: Consider the triangle graph C_3 with the vertices u, v, and w and the edges (u, v), (u, w), and (v, w).



- C_3 has a cycle and therefore it is not a tree.
- Assume C_3 is a bipartite graph $H = (X, Y)$ in which all the edges are between vertices from X and vertices from Y. The existence of the edge (u, v) implies that u and v must belong one to X and one to Y. But then the edges (u, w) and (v, w) imply that w can belong to neither X nor Y. A contradiction.

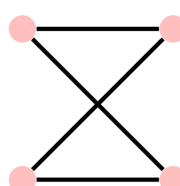
In general, the complete graph K_n for $n \geq 3$ is not a bipartite graph and is not a tree.

Proposition: $\mathcal{T} \subseteq \mathcal{B}$: Every tree is a bipartite graph.

Proof: A tree can be colored with two colors (see Part (c) of Problem 19). Call the colors 1 and 2. Consider the following partition of the vertices of the tree: X contains all the vertices whose color is 1 and Y contains all the vertices whose color is 2. The tree is bipartite since by the definition of coloring there are no edges between vertices with the same color and as a result there are no edges between two vertices from X or between two vertices from Y.

Proposition: $\mathcal{B} \not\subseteq \mathcal{T}$: There exist bipartite graphs that are not trees.

Proof: Bipartite may have cycles (but only even size cycles) while trees cannot have cycles. The smallest example of a bipartite graph that is not a tree is C_4 the cycle of size 4.

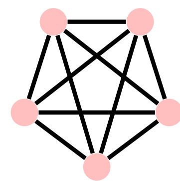


In general, any complete bipartite graph $K_{r,s}$ in which $r, s \geq 2$ is not a tree because it contains at least one cycle.

Corollary: $\mathcal{T} \subset \mathcal{B} \subset \mathcal{G}$.

27. Prove that for every $n \geq 5$ there exists a graph with n vertices, all of which have degree 4.

Proof I: By induction. The base case is $n = 5$. The degree of each vertex in the complete graph K_5 is 4. See the figure below.



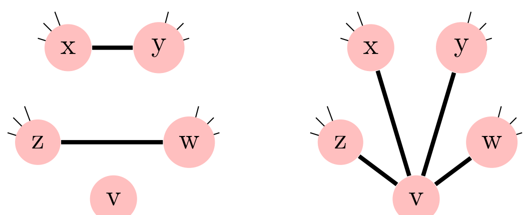
Assume that for $n \geq 6$ there exists a graph G with $n-1$ vertices in which the degrees of all its vertices $v_0, v_1, \ldots, v_{n-2}$ are 4. Therefore, the sum of the degrees of all the vertices of G is $4(n-1)$. By the *hand-shaking lemma*, this sum is twice the number of edges in G. Hence, G has exactly $2n-2$ edges.

Observe that the only graphs in which any pair of edges intersect are the star graphs S_n for $n \geq 3$ and C_3 the cycle of 3. Since G has $2n-2$ edges it cannot be a star which has only $n-1$ edges and since $n-1 > 3$ G cannot be C_3. As a result, G must have two disjoint edges (x, y) and (z, w) for which the four vertices $0 \leq x, y, z, w \leq n-2$ are different.

Let H be the graph G without the edges (x, y) and (z, w) and with a new vertex $v = n-1$ that is connected to the vertices x, y, z, w. By definition the degree of v in H is 4. Also, the degree in H of any vertex that is not x, y, z, w remains 4 as it was in G. Finally, the degrees in H of the four vertices x, y, z, w remain 4 because each one of them lost an edge and gained a new edge with v.

It follows that H has n vertices and that the degrees of all of them are 4 as required.

See below a figure that shows the changes from G (the left side of the figure) to H (the right side of the figure) for the four vertices x, y, z, w and the new vertex v. Note that there could be edges in G and therefore also in H between other pairs of vertices among the four vertices x, y, z, w.



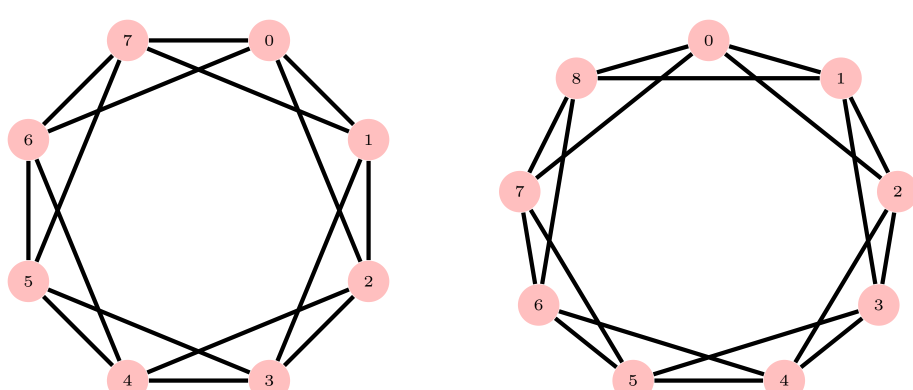
Proof II: For $n \geq 5$, construct the following graph with the vertices $0, 1, \ldots, n-1$. The four neighbors of vertex $0 \leq i \leq n-1$ are: $i-2, i-1, i+1, i+2$ where the additions and subtractions are done modulo n (e.g., $(n-1)+1 = 0$ and $1-2 = n-1$).

For example, for $n \geq 11$, the four neighbors of vertex 8 are vertices $\{6, 7, 9, 10\}$ and the four neighbors of vertex 1 are $\{n-1, 0, 2, 3\}$.

Observe, that the edge definitions are consistent since if $j = i+1$ then $i = j-1$ and if $j = i+2$ then $i = j-2$.

By definition, the degree of each vertex is exactly 4 as required.

See the figure below for the case $n = 8$ on the left side and the case $n = 9$ on the right side.



28. Prove that no graph has all the degrees different; that is, prove that in a degree sequence there is at least one repeated number.

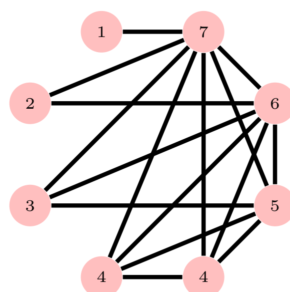
Proof: Assume a graph with $n \geq 2$ vertices. There are only n possible values for a degree of a vertex in a simple graph: $0, 1, \ldots, n-1$. However if the degree of a vertex is 0 (no neighbors) then no other vertex may have degree $n-1$ (being a neighbor of all the other vertices) and if the degree of a vertex is $n-1$ (being a neighbor of all the other vertices) then no other vertex may have degree 0 (no neighbors). As a result there are only two options for values of a degree of a vertex in a simple graph: either $0, 1, \ldots, n-2$ or $1, \ldots, n-1$. In both cases there are only $n-1$ possible values for the degree. By the pigeonhole principle, two out of the n degrees must be equal.

Generalization for an even $n \geq 2$: The only graphic sequence of length n in which all the degrees are different except one degree that appears twice is:

$$(n-1, n-2, \ldots, \frac{n}{2}+2, \frac{n}{2}+1, \frac{n}{2}, \frac{n}{2}, \frac{n}{2}-1, \frac{n}{2}-2, \ldots, 2, 1)$$

Moreover, this sequence has only one non-isomorphic realization.

These realizable sequences for $n = 2, 4, 6$ are $(1,1)$, $(3,2,2,1)$, and $(5,4,3,3,2,1)$ respectively. For $n = 8$, see below the unique realization of the sequence and $(7,6,5,4,4,3,2,1)$.



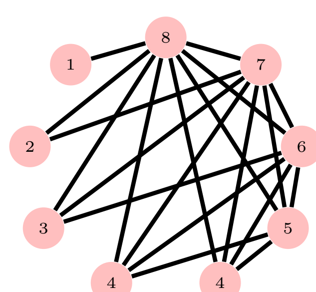
Proof sketch: The vertex with degree $n-1$ fixes the vertex with degree 1, the vertex with degree $n-2$ fixes the vertex with degree 2 that is now connected to the vertex with degree $n-1$ and to the vertex with degree $n-2$. This process continues until there are two vertices with degree $\frac{n}{2}-1$ that are connected to the first $\frac{n}{2}-1$ vertices. After adding an edge between these two vertices, the sequence is realized.

Generalization for an odd $n \geq 3$: The only graphic sequence of length n in which all the degrees are different except one degree that appears twice is:

$$(n-1, n-2, \ldots, \frac{n-1}{2}+2, \frac{n-1}{2}+1, \frac{n-1}{2}, \frac{n-1}{2}, \frac{n-1}{2}-1, \frac{n-1}{2}-2, \ldots, 2, 1)$$

Moreover, this sequence has only one non-isomorphic realization.

These realizable sequences for $n = 3, 5, 7$ are $(2,1,1)$, $(4,3,2,2,1)$, and $(6,5,4,3,3,2,1)$ respectively. For $n = 9$, see below the unique realization of the sequence and $(8,7,6,5,4,4,3,2,1)$.



Proof sketch: The vertex with degree $n-1$ fixes the vertex with degree 1, the vertex with degree $n-2$ fixes the vertex with degree 2 that is now connected to the vertex with degree $n-1$ and to the vertex with degree $n-2$. This process continues until there are two vertices with degree $\frac{n-1}{2}$ that are connected to the first $\frac{n-1}{2}$ vertices.

29. A *bridge* in a connected graph is an edge (u, v) whose removal from the graph disconnects it. Prove that if G is a connected graph for which all the vertices have an even degree, then G does not have a bridge.

Proof: Let G be a connected graph for which all the vertices have an even degree. Assume to the contrary that the edge $e = (u, v)$ is a bridge in G. Remove the e from G. Since e is a bridge, G is partitioned to two disjoint connected components G_u and G_v such that $u \in G_u$ and $v \in G_v$.

The degree of any vertex in G_u except u is even since the removal of e changes only the degrees of u and v. The degree of u in G_u is odd because it is reduced by one after the removal of e. As a result, the sum of the degrees of all the vertices in G_u is odd. A contradiction, since by the *hand-shaking lemma* the sum of the degrees in any graph is even ($2m$ for a graph with m edges).

Proof illustration: Below is a graph with 11 vertices. G_u is the sub-graph composed of the 6 vertices in the left side and G_v is the sub-graph composed of the 5 vertices in the right side. The edge (u, v) is a bridge because if it is removed both G_u and G_v would be disconnected. In this example, the degrees of all the vertices except u and v are even. However, the degrees of both u and v must be odd. This is because otherwise after removing the edge (u, v) the sum of the degrees in both G_u and G_v would be odd contradicting the hand-shaking lemma.