Discrete Math

Induction Practice Problems: Solutions

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1. Prove the following identity by induction on $n \geq 1$.

$$\sum_{i=1}^{n} i(i+1) = 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$$

Proof by induction:

- *Notations.*

$$\begin{aligned} L(n) &= 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) \\ R(n) &= \frac{n(n+1)(n+2)}{3} \end{aligned}$$

- *Induction base.* Prove that $L(1) = R(1)$:

$$L(1) = 1 \cdot 2 = 2 = \frac{1 \cdot 2 \cdot 3}{3} = R(1)$$

- *Induction hypothesis.* Assume that $L(k) = R(k)$ for $k \geq 1$:

$$\begin{aligned} \sum_{i=1}^{k} i(i+1) &= 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + k(k+1) \\ &= \frac{k(k+1)(k+2)}{3} \end{aligned}$$

- *Inductive step.* Prove that $L(k+1) = R(k+1)$ for $k \geq 1$:

$$\begin{aligned} L(k+1) &= 1 \cdot 2 + 2 \cdot 3 + + \cdots + k(k+1) + (k+1)(k+2) && (* \text{ definition of } L(k+1) *) \\ &= L(k) + (k+1)(k+2) && (* \text{ definition of } L(k) *) \\ &= R(k) + (k+1)(k+2) && (* \text{ induction hypothesis } *) \\ &= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) && (* \text{ definition of } R(k) *) \\ &= \frac{k(k+1)(k+2) + 3(k+1)(k+2)}{3} && (* \text{ algebra } *) \\ &= \frac{(k+3)(k+1)(k+2)}{3} && (* \text{ algebra } *) \\ &= \frac{(k+1)(k+2)(k+3)}{3} && (* \text{ algebra } *) \\ &= R(k+1) && (* \text{ definition of } R(k+1) *) \end{aligned}$$

A proof without induction: Recall that

$$\begin{aligned} \sum_{i=1}^{n} i &= \frac{n(n+1)}{2} \\ \sum_{i=1}^{n} i^2 &= \frac{n(n+1)(2n+1)}{6} \end{aligned}$$

These two identities imply the following,

$$\begin{aligned}
L(n) &= \sum_{i=1}^{n} i(i+1) && (* \text{ definition of } L(n) *) \\
&= \sum_{i=1}^{n} (i^2+i) && (* \text{ algebra } *) \\
&= \sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i && (* \text{ algebra } *) \\
&= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} && (* \text{ applying above identities } *) \\
&= \frac{n(n+1)(2n+1)+3n(n+1)}{6} && (* \text{ algebra } *) \\
&= \frac{n(n+1)((2n+1+3))}{6} && (* \text{ algebra } *) \\
&= \frac{n(n+1)(2n+4)}{6} && (* \text{ algebra } *) \\
&= \frac{n(n+1)2(n+2)}{6} && (* \text{ algebra } *) \\
&= \frac{n(n+1)(n+2)}{3} && (* \text{ algebra } *) \\
&= R(n) && (* \text{ definition of } R(n) *)
\end{aligned}$$

Visual proofs:

- `https://www.youtube.com/watch?v=U7sQ4b7DAHg`
- `https://www.youtube.com/watch?v=Hb5NOMoDNH8`
- `https://www.youtube.com/watch?v=kedEuSNQf6I`

2. Prove the following identity by induction on $n \geq 2$.

$$\sum_{i=1}^{n-1} \frac{1}{i(i+1)} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{(n-1)n} = 1 - \frac{1}{n}$$

Proof by induction:

- *Notations.*

$$\begin{aligned} L(n) &= \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{(n-1)n} \\ R(n) &= 1 - \frac{1}{n} \end{aligned}$$

- *Induction base.* Prove that $L(2) = R(2)$:

$$L(2) = \frac{1}{1 \cdot 2} = \frac{1}{2} = 1 - \frac{1}{2} = R(2)$$

- *Induction hypothesis.* Assume that $L(k) = R(k)$ for $k \geq 2$:

$$\begin{aligned} \sum_{i=1}^{k-1} \frac{1}{i(i+1)} &= \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{(k-1)k} \\ &= 1 - \frac{1}{k} \end{aligned}$$

- *Inductive step.* Prove that $L(k+1) = R(k+1)$ for $k \geq 2$:

$$\begin{aligned} L(k+1) &= \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{(k-1)k} + \frac{1}{k(k+1)} && (* \text{ definition of } L(k+1) *) \\ &= L(k) + \frac{1}{k(k+1)} && (* \text{ definition of } L(k) *) \\ &= R(k) + \frac{1}{k(k+1)} && (* \text{ induction hypothesis } *) \\ &= 1 - \frac{1}{k} + \frac{1}{k(k+1)} && (* \text{ definition of } R(k) *) \\ &= 1 - \frac{k+1}{k(k+1)} + \frac{1}{k(k+1)} && (* \text{ algebra } *) \\ &= 1 - \frac{(k+1)-1}{k(k+1)} && (* \text{ algebra } *) \\ &= 1 - \frac{k}{k(k+1)} && (* \text{ algebra } *) \\ &= 1 - \frac{1}{k+1} && (* \text{ algebra } *) \\ &= R(k+1) && (* \text{ definition of } R(n) *) \end{aligned}$$

A proof without induction: The following is an identity for any integer $i \geq 1$,

$$\frac{1}{i} - \frac{1}{i+1} = \frac{(i+1)-i}{i(i+1)} = \frac{1}{i(i+1)}$$

This identity implies the following,

$$\begin{aligned}
L(n) &= \sum_{i=1}^{n-1} \frac{1}{i(i+1)} && \text{(* definition of } L(n) \text{ *)} \\
&= \sum_{i=1}^{n-1} \left(\frac{1}{i} - \frac{1}{i+1}\right) && \text{(* applying above identity *)} \\
&= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n-2} - \frac{1}{n-1}\right) + \left(\frac{1}{n-1} - \frac{1}{n}\right) \\
& && \text{(* opening the sum *)} \\
&= \frac{1}{1} - \left(\frac{1}{2} - \frac{1}{2}\right) - \left(\frac{1}{3} - \frac{1}{3}\right) - \cdots - \left(\frac{1}{n-1} - \frac{1}{n-1}\right) - \frac{1}{n} \\
&= && \text{(* rearranging terms *)} \\
&= 1 - \frac{1}{n} && \text{(* algebra *)} \\
&= R(n) && \text{(* definition of } R(n) \text{ *)}
\end{aligned}$$

3. Prove by induction that $n! > 2^n$ for all integers $n \geq 4$.

The inequality is wrong for $1 \leq n \leq 3$

$$\begin{array}{rcl} 1! = 1 & < & 2^1 = 2 \\ 2! = 2 & < & 2^2 = 4 \\ 3! = 6 & < & 2^3 = 8 \\ 4! = 24 & > & 2^4 = 16 \\ 5! = 120 & > & 2^5 = 32 \\ 6! = 720 & > & 2^6 = 64 \end{array}$$

Proof by induction:

- *Induction base.* $4! = 24 > 16 = 2^4$ for $n = 4$.
- *Induction hypothesis.* Assume that $k! > 2^k$ for $k \geq 4$.
- *Inductive step.* Prove that $(k+1)! > 2^{k+1}$ for $k \geq 4$.

$$\begin{array}{rcll} (k+1)! & = & (k+1)k! & (* \text{ definition of } (k+1)! *) \\ & > & (k+1)2^k & (* \text{ induction hypothesis } *) \\ & > & 2 \cdot 2^k & (* \ k > 1 \Rightarrow (k+1) > 2 *) \\ & = & 2^{k+1} & (* \text{ definition of } 2^{k+1} *) \end{array}$$

4. Prove by induction that $8^n - 1$ is divisible by 7 for all integers $n \geq 1$.

Proof by induction:

- *Induction base.* $8^1 - 1 = 7 = 7 \cdot 1$ for $n = 1$.
- *Induction hypothesis.* Assume that $8^k - 1 = 7 \cdot q$ for an integer q.
- *Inductive step.* Prove that $8^{k+1} - 1 = 7 \cdot p$ for an integer p.

$$\begin{aligned}
8^{k+1} - 1 &= 8 \cdot 8^k - 1 && (* \text{ algebra } *) \\
&= 7 \cdot 8^k + (8^k - 1) && (* \text{ algebra } *) \\
&= 7 \cdot 8^k + 7 \cdot q && (* \text{ induction hypothesis } *) \\
&= 7(8^k + q) && (* \text{ algebra } *)
\end{aligned}$$

Hence, $8^{k+1} = 7 \cdot p$ for integer $p = 8^k + q$ and is therefore divisible by 7.

Another proof: The following is an identity for any positive real numbers $x \geq y$ and an integer $n \geq 1$,

$$x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + x^2y^{n-3} + xy^{n-2} + y^{n-1})$$

For $x = 8$ and $y = 1$ the above identity is equivalent to the following identity,

$$\begin{aligned}
8^n - 1 &= (8 - 1)(8^{n-1} + 8^{n-2} + \cdots + 8 + 1) \\
&= 7(8^{n-1} + 8^{n-2} + \cdots + 8 + 1)
\end{aligned}$$

This implies that $8^n - 1$ is divisible by 7.

5. A ternary string of length n is a list of n digits in which each digit is either 0, or 1, or 2. Prove by induction that there are 3^n ternary strings of length n for all integers $n \geq 1$.

Proof by induction:

- *Induction base.* $3^1 = 3$ and there are 3 ternary strings of length 1: (0), (1), and (2).
- *Induction hypothesis.* Assume that there are 3^k ternary strings of length k for $k \geq 1$.
- *Inductive step.* Prove that there are 3^{k+1} ternary strings of length $k+1$ for $k \geq 1$.

  For any ternary string $s = (d_1 d_2 \cdots d_k)$ of length k in which $d_i \in \{0, 1, 2\}$ for $1 \leq i \leq k$ define the following three extensions into ternary strings of length $k+1$:

  - $s_0 = (d_1 d_2 \cdots d_k 0)$
  - $s_1 = (d_1 d_2 \cdots d_k 1)$
  - $s_2 = (d_1 d_2 \cdots d_k 2)$

  As a result, by the induction hypothesis, there are at least $3 \cdot 3^k = 3^{k+1}$ ternary strings of length $k+1$. Equality holds since any ternary string $s' = (d_1 d_2 \cdots d_k d_{k+1})$ is the $s_{d_{k+1}}$ extension of the ternary string $s = (d_1 d_2 \cdots d_k)$.