Discrete Structures

Probability Practice Problems: Solutions

1. Assume a **fair** coin in which the probability for flipping Heads (H) is $p(H) = 1/2$ and the probability of flipping Tails (T) is $p(T) = 1/2$.

   Proposition I: When the fair coin is flipped $n \geq 1$ times, the probability of flipping $0 \leq h \leq n$ heads and $n - h$ tails is

$$\frac{\binom{n}{h}}{2^n}$$

   Proof: There are 2^n equally probable sequences of n coin flips. There are $\binom{n}{h}$ ways to select the flips that show heads. Therefore, the probability of flipping exactly $0 \leq h \leq n$ heads is $\frac{\binom{n}{h}}{2^n}$.

   Remark: Observe that

$$\sum_{h=0}^{n} \binom{n}{h} = 2^n$$

   and therefore

$$\sum_{h=0}^{n} \frac{\binom{n}{h}}{2^n} = \frac{\sum_{h=0}^{n} \binom{n}{h}}{2^n} = \frac{2^n}{2^n} = 1$$

   (a) The fair coin is flipped 4 times.

      i. What is the probability that **exactly** 2 out of the 4 flips are heads?

         Answer: Apply Proposition I with $n = 4$ and $h = 2$:

$$\frac{\binom{4}{2}}{2^4} = \frac{6}{16} = \frac{3}{8} = 0.375 = 37.5\%$$

      ii. What is the probability that **at least** 2 out of the 4 flips are heads?

         Answer: Apply Proposition I with $n = 4$ and $h = 2, 3, 4$:

$$\frac{\binom{4}{2} + \binom{4}{3} + \binom{4}{4}}{2^4} = \frac{6 + 4 + 1}{16} = \frac{11}{16} = 0.6875 = 68.75\%$$

      iii. What is the probability that **at most** 2 out of the 4 flips are heads?

         Answer: Apply Proposition I with $n = 4$ and $h = 0, 1, 2$:

$$\frac{\binom{4}{0} + \binom{4}{1} + \binom{4}{2}}{2^4} = \frac{1 + 4 + 6}{16} = \frac{11}{16} = 0.6875 = 68.75\%$$

   (b) The fair coin is flipped 5 times.

      i. What is the probability that **exactly** 3 out of the 5 flips are heads?

         Answer: Apply Proposition I with $n = 5$ and $h = 3$:

$$\frac{\binom{5}{3}}{2^5} = \frac{10}{32} = \frac{5}{16} = 0.3125 = 31.25\%$$

ii. What is the probability that **at least** 3 out of the 5 flips are heads?

Answer: Apply Proposition I with $n = 5$ and $h = 3, 4, 5$:

$$\frac{\binom{5}{3} + \binom{5}{4} + \binom{5}{5}}{2^5} = \frac{10 + 5 + 1}{32} = \frac{16}{32} = \frac{1}{2} = 0.5 = 50\%$$

iii. What is the probability that **at most** 3 out of the 5 flips are heads?

Answer: Apply Proposition with $n = 5$ and $h = 0, 1, 2, 3$:

$$\frac{\binom{5}{0} + \binom{5}{1} + \binom{5}{2} + \binom{5}{3}}{2^5} = \frac{1 + 5 + 10 + 10}{32} = \frac{26}{32} = \frac{13}{16} = 0.8125 = 81.25\%$$

(c) The fair coin is flipped $n \geq 2$ times.

i. What is the probability that **exactly** one out of the n flips is heads?

Answer: Apply Proposition I with $h = 1$:

$$\frac{\binom{n}{1}}{2^n} = \frac{n}{2^n}$$

ii. What is the probability that **at least** one out of the n flips is heads?

Answer: Proposition I implies that the probability of flipping only tails ($h = 0$) is

$$\frac{\binom{n}{0}}{2^n} = \frac{1}{2^n}$$

The event that at least one out of the n flips is heads is the complement of the event that there are no heads. Therefore, the probability of this event is

$$1 - \frac{1}{2^n} = \frac{2^n}{2^n} - \frac{1}{2^n} = \frac{2^n - 1}{2^n}$$

iii. What is the probability that **at most** one out of the n flips is heads?

Answer: Apply Proposition I with $h = 0$ and $h = 1$:

$$\frac{\binom{n}{0} + \binom{n}{1}}{2^n} = \frac{1 + n}{2^n}$$

2. Assume a **biased** coin in which the probability for flipping Head (H) is $p(H) = 2/3$ and the probability of flipping Tail (T) is $p(T) = 1/3$.

Proposition II: When the biased coin is flipped $n \geq 1$ times, the probability of flipping exactly $0 \leq h \leq n$ heads and $n - h$ tails is

$$\frac{\binom{n}{h} \cdot 2^h}{3^n}$$

Proof: There are 2^n sequences of n coin flips. Since $p(H) = 2/3$ and $p(T) = 1/3$, it follows that the probability of a sequence with h heads and $n - h$ tails is

$$\left(\frac{2}{3}\right)^h \cdot \left(\frac{1}{3}\right)^{n-h} = \frac{2^h}{3^h} \cdot \frac{1^{n-h}}{3^{n-h}} = \frac{2^h \cdot 1^{n-h}}{3^h \cdot 3^{n-h}} = \frac{2^h}{3^n}$$

There are $\binom{n}{h}$ ways to select the flips that show heads. Therefore, the probability of flipping exactly $0 \leq h \leq n$ heads is $\frac{\binom{n}{h} \cdot 2^h}{3^n}$.

Remark: Recall that $(x + y)^n = \sum_{h=0}^{n} \binom{n}{h} x^{n-h} \cdot y^h$. Consider the case in which $x = 1$ and $y = 2$

$$3^n = (1 + 2)^n = \sum_{h=0}^{n} \binom{n}{h} 1^{n-h} \cdot 2^h = \sum_{h=0}^{n} \binom{n}{h} 2^h$$

and therefore

$$\sum_{h=0}^{n} \frac{\binom{n}{h} \cdot 2^h}{3^n} = \frac{\sum_{h=0}^{n} \left(\binom{n}{h} \cdot 2^h\right)}{3^n} = \frac{3^n}{3^n} = 1$$

(a) The biased coin is flipped 4 times.

i. What is the probability that **exactly** 2 out of the 4 flips are heads?

Answer: Apply Proposition II with $n = 4$ and $h = 2$:

$$\frac{\binom{4}{2} \cdot 2^2}{3^4} = \frac{6 \cdot 4}{81} = \frac{24}{81} = \frac{8}{27} = 0.296296\ldots \approx 29.63\%$$

ii. What is the probability that **at least** 2 out of the 4 flips are heads?

Answer: Apply Proposition II with $n = 4$ and $h = 2, 3, 4$:

$$\frac{\binom{4}{2} \cdot 2^2 + \binom{4}{3} \cdot 2^3 + \binom{4}{4} \cdot 2^4}{3^4} = \frac{6 \cdot 4 + 4 \cdot 8 + 1 \cdot 16}{81}$$

$$= \frac{24 + 32 + 16}{81} = \frac{72}{81} = \frac{8}{9} = 0.888\ldots \approx 88.89\%$$

iii. What is the probability that **at most** 2 out of the 4 flips are heads?

Answer: Apply Proposition II with $n = 4$ and $h = 0, 1, 2$:

$$\frac{\binom{4}{0} \cdot 2^0 + \binom{4}{1} \cdot 2^1 + \binom{4}{2} \cdot 2^2}{3^4} = \frac{1 \cdot 1 + 4 \cdot 2 + 6 \cdot 4}{16}$$

$$= \frac{1 + 8 + 24}{81} = \frac{33}{81} = \frac{11}{27} = 0.407407\ldots \approx 40.74\%$$

(b) The biased coin is flipped 5 times.

i. What is the probability that **exactly** 3 out of the 5 flips are heads?

Answer: Apply Proposition II with $n = 5$ and $h = 3$:

$$\frac{\binom{5}{3} \cdot 2^3}{3^5} = \frac{10 \cdot 8}{243} = \frac{80}{243} = 0.3292181\ldots \approx 32.92\%$$

ii. What is the probability that **at least** 3 out of the 5 flips are heads?

Answer: Apply Proposition II with $n = 5$ and $h = 3, 4, 5$:

$$\frac{\binom{5}{3} \cdot 2^3 + \binom{5}{4} \cdot 2^4 + \binom{5}{5} \cdot 2^5}{3^5} = \frac{10 \cdot 8 + 5 \cdot 16 + 1 \cdot 32}{243}$$

$$= \frac{80 + 80 + 32}{243} = \frac{192}{243} = \frac{64}{81} = 0.7901234567\ldots \approx 79.01\%$$

iii. What is the probability that **at most** 3 out of the 5 flips are heads?

Answer: Apply Proposition II with $n = 5$ and $h = 0, 1, 2, 3$:

$$\frac{\binom{5}{0} \cdot 2^0 + \binom{5}{1} \cdot 2^1 + \binom{5}{2} \cdot 2^2 + \binom{5}{3} \cdot 2^3}{3^5} = \frac{1 \cdot 1 + 5 \cdot 2 + 10 \cdot 4 + 10 \cdot 8}{243}$$

$$= \frac{1 + 10 + 40 + 80}{243} = \frac{131}{243} = 0.53909465\ldots \approx 53.91\%$$

(c) The biased coin is flipped $n \geq 2$ times.

i. What is the probability that **exactly** one out of the n flips is heads?

Answer: Apply Proposition II with $h = 1$:

$$\frac{\binom{n}{1} \cdot 2^1}{3^n} = \frac{n \cdot 2}{3^n} = \frac{2n}{3^n}$$

ii. What is the probability that **at least** one out of the n flips is heads?

Answer: Proposition II implies that the probability of flipping only tails ($h = 0$) is

$$\frac{\binom{n}{0} \cdot 2^0}{3^n} = \frac{1 \cdot 1}{3^n} = \frac{1}{3^n}$$

The event that at least one out of the n flips is head is the complement of the event that there are no heads. Therefore, the probability of this event is

$$1 - \frac{1}{3^n} = \frac{3^n}{3^n} - \frac{1}{3^n} = \frac{3^n - 1}{3^n}$$

iii. What is the probability that **at most** one out of the n flips is heads?

Answer: Apply Proposition II with $h = 0$ and $h = 1$:

$$\frac{\binom{n}{0} \cdot 2^0 + \binom{n}{1} \cdot 2^1}{3^n} = \frac{1 \cdot 1 + n \cdot 2}{3^n} = \frac{1 + 2n}{3^n}$$

3. What is the probability that there are no consecutive heads and no consecutive tails? By definition, when flipping the fair coin $p(H) = p(T) = 1/2$ and when flipping the biased coin $p(H) = 2/3$ and $p(T) = 1/3$.

   (a) The fair coin is flipped 4 times.

   Answer: $1/8 = 0.125 = 12.5\%$

   Proof: There are 2^4 equally probable sequences of 4 coin flips. In only two of them, $HTHT$ and $THTH$, there are no consecutive heads and no consecutive tails. Therefore, the probability that there are no consecutive heads and no consecutive tails is

$$\frac{2}{2^4} = \frac{2}{16} = \frac{1}{8}$$

   (b) The fair coin is flipped 5 times.

   Answer: $1/16 = 0.0625 = 6.25\%$

   Proof: There are 2^5 equally probable sequences of 5 coin flips. In only two of them, $HTHTH$ and $THTHT$, there are no consecutive heads and no consecutive tails. Therefore, the probability that there are no consecutive heads and no consecutive tails is

$$\frac{2}{2^5} = \frac{2}{32} = \frac{1}{16}$$

   (c) The fair coin is flipped $n \geq 1$ times.

   Answer: $\frac{1}{2^{n-1}}$

   Proof: There are 2^n equally probable sequences of n coin flips. In only two of them, $HTHT\ldots$ and $THTH\ldots$, there are no consecutive heads and no consecutive tails. Therefore, the probability that there are no consecutive heads and no consecutive tails is

$$\frac{2}{2^n} = \frac{1}{2^{n-1}}$$

   (d) The biased coin is flipped 4 times.

   Answer: $\frac{8}{81} = 0.098765432\ldots \approx 9.88\%$

   Proof: There are 2^4 different sequences of 4 coin flips. If a sequence has h heads and $4-h$ tails, then the probability of this sequence to be the outcome of the 4 flips is

$$\left(\frac{2}{3}\right)^h \left(\frac{1}{3}\right)^{(4-h)} = \frac{2^h}{3^h} \cdot \frac{1^{4-h}}{3^{4-h}} = \frac{2^h \cdot 1^{4-h}}{3^h \cdot 3^{4-h}} = \frac{2^h}{3^4}$$

   In only two of the sequences, $HTHT$ and $THTH$, there are no consecutive heads and no consecutive tails. In both sequences $h = 2$. Therefore, the probability that there are no consecutive heads and no consecutive tails is

$$2 \cdot \frac{2^2}{3^4} = 2 \cdot \frac{4}{81} = \frac{8}{81}$$

(e) The biased coin is flipped 5 times.

Answer: $\frac{4}{81} = 0.0493827\ldots \approx 4.94\%$

Proof: There are 2^5 different sequences of 5 coin flips. If a sequence has h heads and $5-h$ tails, then the probability of this sequence to be the outcome of the 5 flips is

$$\left(\frac{2}{3}\right)^h \left(\frac{1}{3}\right)^{(5-h)} = \frac{2^h}{3^h} \cdot \frac{1^{5-h}}{3^{5-h}} = \frac{2^h \cdot 1^{5-h}}{3^k \cdot 3^{5-h}} = \frac{2^h}{3^5}$$

In only two of the sequences, $HTHTH$ and $THTHT$, there are no consecutive heads and no consecutive tails. In the first sequence $h = 3$ while in the second sequence $h = 2$. Therefore, the probability that there are no consecutive heads and no consecutive tails is

$$\frac{2^3}{3^5} + \frac{2^2}{3^5} = \frac{2^3 + 2^2}{3^5} = \frac{8+4}{243} = \frac{12}{243} = \frac{4}{81}$$

(f) The biased coin is flipped $n \geq 1$ times.

Answer:

- $2 \cdot \left(\frac{2}{9}\right)^{n/2}$ for an even $n \geq 2$.
- $\left(\frac{2}{9}\right)^{(n-1)/2}$ for an odd $n \geq 1$.

Proof: There are 2^n different sequences of n coin flips. If a sequence has h heads and $n-h$ tails, then the probability of this sequence to be the outcome of the n flips is

$$\left(\frac{2}{3}\right)^h \left(\frac{1}{3}\right)^{(n-h)} = \frac{2^h}{3^h} \cdot \frac{1^{n-h}}{3^{n-h}} = \frac{2^h \cdot 1^{n-h}}{3^h \cdot 3^{n-h}} = \frac{2^h}{3^n}$$

For an even $n \geq 2$, the only sequences with no consecutive heads and no consecutive tails are $HT \ldots HT$ and $TH \ldots TH$. In both cases $h = n/2$. Therefore, the probability that there are no consecutive heads and no consecutive tails is

$$2 \cdot \frac{2^{n/2}}{3^n} = 2 \cdot \frac{2^{n/2}}{(3^2)^{n/2}} = 2 \cdot \frac{2^{n/2}}{9^{n/2}} = 2 \cdot \left(\frac{2}{9}\right)^{n/2}$$

For an odd $n \geq 1$, the only sequences with no consecutive heads and no consecutive tails are $HTH \ldots TH$ and $THT \ldots HT$. In the first case $h = (n+1)/2$ and in the second case $h = (n-1)/2$. Therefore the probability that there are no consecutive heads and no consecutive tails is

$$\frac{2^{(n+1)/2}}{3^n} + \frac{2^{(n-1)/2}}{3^n} = \frac{2^{(n+1)/2} + 2^{(n-1)/2}}{3^n} = \frac{2 \cdot 2^{(n-1)/2} + 2^{(n-1)/2}}{3 \cdot 3^{n-1}}$$

$$= \frac{3 \cdot 2^{(n-1)/2}}{3 \cdot (3^2)^{(n-1)/2}} = \frac{2^{(n-1)/2}}{9^{(n-1)/2}} = \left(\frac{2}{9}\right)^{(n-1)/2}$$

4. In a fair 6-face dice, with the numbers $1, 2, 3, 4, 5, 6$ on its faces, the probability of throwing any of the 6 numbers is $1/6$.

   (a) What is the probability of throwing exactly two 6 when three dice are thrown together?

   Answer: $\frac{5}{72} = 0.06944\ldots \approx 6.94\%$

   Proof: There are 6^3 different combinations for the three dice when order matters. There are $\binom{3}{2}$ ways to have two dice that show 6. The dice that does not show 6 may show any of the five numbers $1, 2, 3, 4, 5$. Therefore, the probability of throwing exactly two 6 when three dice are thrown together is

$$\frac{\binom{3}{2} \cdot 5}{6^3} = \frac{3 \cdot 5}{216} = \frac{5}{72}$$

   (b) What is the probability of throwing exactly two 6 when four dice are thrown together?

   Answer: $\frac{25}{216} = 0.11574\ldots \approx 11.58\%$

   Proof: There are 6^4 different combinations for the four dice when order matters. There are $\binom{4}{2}$ ways to have two dice that show 6. The two dice that do not show 6 may show any of the five numbers $1, 2, 3, 4, 5$. Therefore, the probability of throwing exactly two 6 when four dice are thrown together is

$$\frac{\binom{4}{2} \cdot 5^2}{6^4} = \frac{6 \cdot 25}{1296} = \frac{25}{216}$$

   (c) What is the probability of throwing exactly two 6 when $n \geq 2$ dice are thrown together?

   Answer: $\frac{\binom{n}{2} \cdot 5^{n-2}}{6^n}$

   Proof: There are 6^n different combinations for the n dice when order matters. There are $\binom{n}{2}$ ways to have two dice that show 6. The $n-2$ dice that do not show 6 may show any of the five numbers $1, 2, 3, 4, 5$. Therefore, the probability of throwing exactly two 6 when n dice are thrown together is

$$\frac{\binom{n}{2} \cdot 5^{n-2}}{6^n} = \frac{\frac{n(n-1)}{2} \cdot \frac{5^n}{25}}{6^n} = \frac{n(n-1)}{50} \left(\frac{5}{6}\right)^n$$

(d) What is the probability of throwing at least one 6 when three dice are thrown together?

Answer: $\frac{91}{216} = 0.421296\ldots \approx 42.13\%$

Proof: The probability of not throwing a 6 when throwing one dice is $\frac{5}{6}$ and when throwing three dice it is $\left(\frac{5}{6}\right)^3$. The complement event of the latter is the event of throwing at least one 6 when three dice are thrown together. Therefore, the probability of this event is

$$1 - \left(\frac{5}{6}\right)^3 = \frac{6^3}{6^3} - \frac{5^3}{6^3} = \frac{6^3 - 5^3}{6^3} = \frac{216 - 125}{216} = \frac{91}{216}$$

(e) What is the probability of throwing at least one 6 when four dice are thrown together?

Answer: $\frac{671}{1296} = 0.5177469\ldots \approx 51.77\%$

Proof: The probability of not throwing a 6 when throwing one dice is $\frac{5}{6}$ and when throwing four dice it is $\left(\frac{5}{6}\right)^4$. The complement event of the latter is the event of throwing at least one 6 when four dice are thrown together. Therefore, the probability of this event is

$$1 - \left(\frac{5}{6}\right)^4 = \frac{6^4}{6^4} - \frac{5^4}{6^4} = \frac{6^4 - 5^4}{6^4} = \frac{1296 - 625}{1296} = \frac{671}{1296}$$

(f) What is the probability of throwing at least one 6 when $n \geq 1$ dice are thrown together?

Answer: $\frac{6^n - 5^n}{6^n}$

Proof: The probability of not throwing a 6 when throwing one dice is $\frac{5}{6}$ and when throwing n dice it is $\left(\frac{5}{6}\right)^n$. The complement event of the latter is the event of throwing at least one 6 when n dice are thrown together. Therefore, the probability of this event is

$$1 - \left(\frac{5}{6}\right)^n = \frac{6^n}{6^n} - \frac{5^n}{6^n} = \frac{6^n - 5^n}{6^n}$$

5. In a fair 5-face dice, with the numbers $1, 2, 3, 4, 5$ on its faces, the probability of throwing any of the 5 numbers is $1/5$.

   (a) What is the probability of throwing exactly two 5 when three dice are thrown together?

   Answer: $\frac{12}{125} = 0096 \approx 10\%$

   Proof: There are 5^3 different combinations for the three dice when order matters. There are $\binom{3}{2}$ ways to have two dice that show 5. The dice that does not show 5 may show any of the four numbers $1, 2, 3, 4$. Therefore, the probability of throwing exactly two 5 when three dice are thrown together is

$$\frac{\binom{3}{2} \cdot 4}{5^3} = \frac{3 \cdot 4}{125} = \frac{12}{125}$$

   (b) What is the probability of throwing exactly two 5 when four dice are thrown together?

   Answer: $\frac{96}{625} = 0.1536 = 15.36\%$

   Proof: There are 5^4 different combinations for the four dice when order matters. There are $\binom{4}{2}$ ways to have two dice that show 5. The two dice that do not show 5 may show any of the four numbers $1, 2, 3, 4$. Therefore, the probability of throwing exactly two 5 when four dice are thrown together is

$$\frac{\binom{4}{2} \cdot 4^2}{5^4} = \frac{6 \cdot 16}{625} = \frac{96}{625}$$

   (c) What is the probability of throwing exactly two 5 when $n \geq 2$ dice are thrown together?

   Answer: $\frac{\binom{n}{2} \cdot 4^{n-2}}{5^n}$

   Proof: There are 5^n different combinations for the n dice when order matters. There are $\binom{n}{2}$ ways to have two dice that show 5. The $n-2$ dice that do not show 5 may show any of the four numbers $1, 2, 3, 4$. Therefore, the probability of throwing exactly two 5 when n dice are thrown together is

$$\frac{\binom{n}{2} \cdot 4^{n-2}}{5^n} = \frac{\frac{n(n-1)}{2} \cdot \frac{4^n}{16}}{5^n} = \frac{n(n-1)}{32} \left(\frac{4}{5}\right)^n$$

(d) What is the probability of throwing at least one 5 when three dice are thrown together?

Answer: $\frac{61}{125} = 0.488 = 48.8\%$

Proof: The probability of not throwing a 5 when throwing one dice is $\frac{4}{5}$ and when throwing three dice it is $\left(\frac{4}{5}\right)^3$. The complement event of the latter is the event of throwing at least one 5 when three dice are thrown together. Therefore, the probability of this event is

$$1 - \left(\frac{4}{5}\right)^3 = \frac{5^3}{5^3} - \frac{4^3}{5^3} = \frac{5^3 - 4^3}{5^3} = \frac{125 - 64}{125} = \frac{61}{125}$$

(e) What is the probability of throwing at least one 5 when four dice are thrown together?

Answer: $\frac{369}{625} = 0.5904 = 59.04\%$

Proof: The probability of not throwing a 5 when throwing one dice is $\frac{4}{5}$ and when throwing four dice it is $\left(\frac{4}{5}\right)^4$. The complement event of the latter is the event of throwing at least one 5 when four dice are thrown together. Therefore, the probability of this event is

$$1 - \left(\frac{4}{5}\right)^4 = \frac{5^4}{5^4} - \frac{4^4}{5^4} = \frac{5^4 - 4^4}{5^4} = \frac{625 - 256}{625} = \frac{369}{625}$$

(f) What is the probability of throwing at least one 5 when $n \geq 1$ dice are thrown together?

Answer: $\frac{5^n - 4^n}{5^n}$

Proof: The probability of not throwing a 5 when throwing one dice is $\frac{4}{5}$ and when throwing n dice it is $\left(\frac{4}{5}\right)^n$. The complement event of the latter is the event of throwing at least one 5 when n dice are thrown together. Therefore, the probability of this event is

$$1 - \left(\frac{4}{5}\right)^n = \frac{5^n}{5^n} - \frac{4^n}{5^n} = \frac{5^n - 4^n}{5^n}$$

6. A deck of cards contains 52 cards. There are 4 suits: 13 Black Clubs, 13 Red Diamonds, 13 Red Hearts, and 13 Black Spades. Each suit has one of the following 9 number cards: $2, 3, 4, 5, 6, 7, 8, 9, 10$, one of the following 3 face cards: Jack (J), Queen (Q), King (K), and one Ace (A). A bridge hand has 13 random cards out of the 52 cards

   Observation: There are $\binom{52}{13} = 635013559600$ different bridge hands.

   (a) What is the probability that all the 13 cards in a bridge hand are of the same suit?

   Answer: $\frac{1}{158753389900} \approx 6.3 \cdot 10^{-12}$

   Proof: There are only 4 bridge hands in which all the cards are of the same suit: (i) all of the 13 cards are clubs, (ii) all of the 13 cards are diamonds, (iii) all of the 13 cards are hearts, and (iv) all of the 13 cards are spades. Therefore, the probability that all the 13 cards in a bridge hand are of the same suit is

$$\frac{4}{\binom{52}{13}} = \frac{4}{635013559600} = \frac{1}{158753389900}$$

   (b) What is the probability that all the 13 cards in a bridge hand are red?

   Answer: $\frac{19}{1160054} = 0.0000163785\ldots \approx \frac{1}{61055}$

   Proof: There are $\binom{26}{13}$ bridge hands in which all the cards are red because a deck contains 26 red cards and 26 black cards. Therefore, the probability that all the 13 cards in a bridge hand are red is

$$\frac{\binom{26}{13}}{\binom{52}{13}} = \frac{10400600}{635013559600} = \frac{19}{1160054}$$

   (c) What is the probability that a bridge hand contains only faces and aces?

   Answer: $\frac{1}{1133952785} \approx 8.8 \cdot 10^{-10}$

   Proof: There are 16 cards out of the 52 that are faces or aces. Hence, there are $\binom{16}{13}$ bridge hands in which all the cards are faces or aces. Therefore, the probability that all the 13 cards in a bridge hand are faces or aces is

$$\frac{\binom{16}{13}}{\binom{52}{13}} = \frac{560}{635013559600} = \frac{1}{1133952785}$$

(d) What is the probability that a bridge hand contains only number cards 2 to 10?

Answer: $\frac{48546}{13340621} = 0.003638961\ldots \approx \frac{1}{275}$

Proof: There are 36 cards out of the 52 that are not faces or aces. Hence, there are $\binom{36}{13}$ bridge hands in which all the cards are number cards 2 to 10. Therefore, the probability that all the 13 cards in a bridge hand are number cards 2 to 10 is

$$\frac{\binom{36}{13}}{\binom{52}{13}} = \frac{2310789600}{635013559600} = \frac{48546}{13340621}$$

(e) What is the probability that a bridge hand does not contain an ace?

Answer: $\frac{6327}{20825} = 0.3038\ldots \approx \frac{3}{10}$

Proof: There are 48 cards out of the 52 that are not aces. Hence, there are $\binom{48}{13}$ bridge hands in which all the cards are not aces. Therefore, the probability that all the 13 cards in a bridge hand are not aces is

$$\frac{\binom{48}{13}}{\binom{52}{13}} = \frac{\frac{48!}{(35!)(13!)}}{\frac{52!}{(39!)(13!)}} = \frac{(48!)(39!)(13!)}{(52!)(35!)(13!)} = \frac{(48!)(39!)}{(52!)(35!)} = \frac{39 \cdot 38 \cdot 37 \cdot 36}{52 \cdot 51 \cdot 50 \cdot 49} = \frac{3 \cdot 19 \cdot 37 \cdot 12}{4 \cdot 17 \cdot 25 \cdot 49} = \frac{6327}{20825}$$

(f) What is the probability that a bridge hand contains all 4 aces?

Answer: $\frac{11}{4165} = 0.002541\ldots \approx \frac{1}{379}$

Proof: There are $\binom{48}{9}$ ways to add 9 cards to a bridge hand that already contains all 4 aces. Therefore, the probability that a bridge hand contains all 4 aces is

$$\frac{\binom{48}{9}}{\binom{52}{13}} = \frac{\frac{48!}{(39!)(9!)}}{\frac{52!}{(39!)(13!)}} = \frac{(48!)(39!)(13!)}{(52!)(39!)(9!)} = \frac{(48!)(13!)}{(52!)(9!)} = \frac{13 \cdot 12 \cdot 11 \cdot 10}{52 \cdot 51 \cdot 50 \cdot 49} = \frac{11}{17 \cdot 5 \cdot 49} = \frac{11}{4165}$$

(g) What is the probability that a bridge hand does not contain two of a kind?

Answer: $\frac{4194304}{39688347475} = 0.0001\ldots \approx \frac{1}{10000}$

Proof: A bridge hand that does not contain two of a kind must contain exactly one card from each of the 13 card-values $2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A$. There are 4 ways to select each of these card-values and 4^{13} ways to select all 13 cards. Therefore, the probability that a bridge hand does not contain two of a kind is

$$\frac{4^{13}}{\binom{52}{13}} = \frac{67108864}{635013559600} = \frac{4194304}{39688347475}$$

7. A bag contains 16 marbles: 3 blue marbles, 5 red marbles, and 8 green marbles.

   (a) After drawing a marble from the bag the marble is put aside.

      i. What is the probability that 2 drawn marbles are of the same color?

      Answer: $\frac{41}{120} = 0.34166\ldots \approx 34.17\%$

      Proof I: There are $\binom{16}{2}$ ways to draw 2 marbles, there are $\binom{3}{2}$ ways to draw 2 blue marbles, there are $\binom{5}{2}$ ways to draw 2 red marbles, and there are $\binom{8}{2}$ ways to draw 2 green marbles. Therefore, the probability of drawing 2 marbles of the same color is

$$\frac{\binom{3}{2}+\binom{5}{2}+\binom{8}{2}}{\binom{16}{2}} = \frac{3+10+28}{120} = \frac{41}{120}$$

      Proof II: There are three cases: (i) with probability 3/16 the first marble is blue and then with probability 2/15 the second marble is also blue; (ii) with probability 5/16 the first marble is red and then with probability 4/15 the second marble is also red; (iii) with probability 8/16 the first marble is green and then with probability 7/15 the second marble is also green. Therefore, the probability of drawing 2 marbles of the same color is

$$\frac{3}{16}\cdot\frac{2}{15}+\frac{5}{16}\cdot\frac{4}{15}+\frac{8}{16}\cdot\frac{7}{15} = \frac{6+20+56}{16\cdot 15} = \frac{82}{240} = \frac{41}{120}$$

      ii. What is the probability that 2 drawn marbles are of different colors?

      Answer: $\frac{79}{120} = 0.65833\ldots \approx 65.83\%$

      Proof I: There are $\binom{16}{2}$ ways to draw 2 marbles, there are $3 \cdot 5$ ways to draw 2 marbles one of them blue and one of them red, there are $3 \cdot 8$ ways to draw 2 marbles one of them blue and one of them green, and there are $5 \cdot 8$ ways to draw 2 marbles one of them red and one of them green. Therefore, the probability of drawing 2 marbles of different colors is

$$\frac{3\cdot 5+3\cdot 8+5\cdot 8}{\binom{16}{2}} = \frac{15+24+40}{120} = \frac{79}{120}$$

      Proof II: There are three cases: (i) with probability 3/16 the first marble is blue and then with probability 13/15 the second marble is red or green; (ii) with probability 5/16 the first marble is red and then with probability 11/15 the second marble is blue or green; (iii) with probability 8/16 the first marble is green and then with probability 8/15 the second marble is blue or red. Therefore, the probability of drawing 2 marbles of different colors is

$$\frac{3}{16}\cdot\frac{13}{15}+\frac{5}{16}\cdot\frac{11}{15}+\frac{8}{16}\cdot\frac{8}{15} = \frac{39+55+64}{16\cdot 15} = \frac{158}{240} = \frac{79}{120}$$

iii. What is the probability that 3 drawn marbles are of the same color?

Answer: $\frac{67}{560} = 0.11964\ldots \approx 11.96\%$

Proof I: There are $\binom{16}{3}$ ways to draw 3 marbles, there are $\binom{3}{3}$ ways to draw 3 blue marbles, there are $\binom{5}{3}$ ways to draw 3 red marbles, and there are $\binom{8}{3}$ ways to draw 3 green marbles. Therefore, the probability of drawing 3 marbles of the same color is

$$\frac{\binom{3}{3}+\binom{5}{3}+\binom{8}{3}}{\binom{16}{3}} = \frac{1+10+56}{560} = \frac{67}{560}$$

Proof II: There are three cases: (i) with probability 3/16 the first marble is blue and then with probability 2/15 the second marble is also blue and then with probability 1/14 the third marble is also blue; (ii) with probability 5/16 the first marble is red and then with probability 4/15 the second marble is also red and then with probability 3/14 the third marble is also red; (iii) with probability 8/16 the first marble is green and then with probability 7/15 the second marble is also green and then with probability 6/14 the third marble is also green. Therefore, the probability of drawing 3 marbles of the same color is

$$\frac{3}{16}\cdot\frac{2}{15}\cdot\frac{1}{14}+\frac{5}{16}\cdot\frac{4}{15}\cdot\frac{3}{14}+\frac{8}{16}\cdot\frac{7}{15}\cdot\frac{6}{14} = \frac{6+60+336}{16\cdot 15\cdot 14} = \frac{402}{3360} = \frac{67}{560}$$

iv. What is the probability that 3 drawn marbles are of different colors?

Answer: $\frac{3}{14} = 0.21428\ldots \approx 21.43\%$

Proof I: There are $\binom{16}{3}$ ways to draw 3 marbles and there are $3\cdot 5\cdot 8$ ways to draw 3 marbles one of them blue, one of them red, and one of them green. Therefore, the probability of drawing 3 marbles of different colors is

$$\frac{3\cdot 5\cdot 8}{\binom{16}{3}} = \frac{120}{560} = \frac{3}{14}$$

Proof II: With probability 3/16 the first marble is blue and then with probability 5/15 the second marble is red and then with probability 8/14 the third marble is green. Hence, with probability

$$\frac{3}{16}\cdot\frac{5}{15}\cdot\frac{8}{14} = \frac{3\cdot 5\cdot 8}{16\cdot 15\cdot 14} = \frac{120}{3360} = \frac{1}{28}$$

the first marble is blue, the second marble is red, and the third marble is green. Similarly, with probability 1/28 the color of the first marble is X, the color of the second marble is Y, and the color of the third marble is Z for any one of the 6 possible assignments of different colors out of blue, red and green to X, Y, and Z. Therefore, the probability of drawing 3 marbles of different colors is

$$6\cdot\frac{1}{28} = \frac{6}{28} = \frac{3}{14}$$

(b) After drawing a marble from the bag the marble is put back in the bag.

i. What is the probability that 2 drawn marbles are of the same color?

Answer: $\frac{49}{128} = 0.3828125 \approx 38.28\%$

Proof I: There are 16^2 ways to draw 2 marbles, there are 3^2 ways to draw 2 blue marbles, there are 5^2 ways to draw 2 red marbles, and there are 8^2 ways to draw 2 green marbles. Therefore, the probability of drawing 2 marbles of the same color is

$$\frac{3^2 + 5^2 + 8^2}{16^2} = \frac{9 + 25 + 64}{256} = \frac{98}{256} = \frac{49}{128}$$

Proof II: With probability $(3/16)^2$ the 2 marbles are blue, with probability $(5/16)^2$ the 2 marbles are red, and with probability $(8/16)^2$ the 2 marbles are green. Therefore, the probability of drawing 2 marbles of the same color is

$$\frac{3^2}{16^2} + \frac{5^2}{16^2} + \frac{8^2}{16^2} = \frac{9 + 25 + 64}{16^2} = \frac{98}{256} = \frac{49}{128}$$

ii. What is the probability that 2 drawn marbles are of different colors?

Answer: $\frac{79}{128} = 0.6171875 \approx 61.72\%$

Proof I: There are 16^2 ways to draw 2 marbles. There are six cases for the 2 marbles to be of different colors: (i) $3 \cdot 5$ ways to draw a blue marble and then a red marble; (ii) $5 \cdot 3$ ways to draw a red marble and then a blue marble; (iii) $3 \cdot 8$ ways to draw a blue marble and then a green marble; (iv) $8 \cdot 3$ ways to draw a green marble and then a blue marble; (v) $5 \cdot 8$ ways to draw a red marble and then a green marble; (vi) $8 \cdot 5$ ways to draw a green marble and then a red marble Therefore, the probability of drawing 2 marbles of different colors is

$$\frac{3 \cdot 5 + 5 \cdot 3 + 3 \cdot 8 + 8 \cdot 3 + 5 \cdot 8 + 8 \cdot 5}{16^2} = \frac{2 \cdot 15 + 2 \cdot 24 + 2 \cdot 40}{256} = \frac{158}{256} = \frac{79}{128}$$

Proof II: There are three cases: (i) with probability $3/16$ the first marble is blue and then with probability $13/16$ the second marble is red or green; (ii) with probability $5/16$ the first marble is red and then with probability $11/16$ the second marble is blue or green; (iii) with probability $8/16$ the first marble is green and then with probability $8/16$ the second marble is blue or red. Therefore, the probability of drawing 2 marbles of different colors is

$$\frac{3}{16} \cdot \frac{13}{16} + \frac{5}{16} \cdot \frac{11}{16} + \frac{8}{16} \cdot \frac{8}{16} = \frac{39 + 55 + 64}{16^2} = \frac{158}{256} = \frac{79}{128}$$

iii. What is the probability that 3 drawn marbles are of the same color?

Answer: $\frac{83}{512} = 0.162109375 \approx 16.21\%$

Proof I: There are 16^3 ways to draw 3 marbles, there are 3^3 ways to draw 3 blue marbles, there are 5^3 ways to draw 3 red marbles, and there are 8^3 ways to draw 3 green marbles. Therefore, the probability of drawing 3 marbles of the same color is

$$\frac{3^3 + 5^3 + 8^3}{16^3} = \frac{27 + 125 + 512}{4096} = \frac{664}{4096} = \frac{83}{512}$$

Proof II: With probability $(3/16)^3$ the 3 marbles are blue, with probability $(5/16)^3$ the 3 marbles are red, and with probability $(8/16)^3$ the 3 marbles are green. Therefore, the probability of drawing 3 marbles of the same color is

$$\frac{3^3}{16^3} + \frac{5^3}{16^3} + \frac{8^3}{16^3} = \frac{27 + 125 + 512}{16^3} = \frac{664}{4096} = \frac{83}{512}$$

iv. What is the probability that 3 drawn marbles are of different colors?

Answer: $\frac{45}{256} = 0.17578125 \approx 17.58\%$

Proof I: There are 16^3 ways to draw 3 marbles, there are $3 \cdot 5 \cdot 8$ ways to draw 3 marbles of different colors for any of the 6 permutations of the colors blue, red, and green assigned to the first, second, and third marbles respectively. Therefore, the probability of drawing 3 marbles of different colors is

$$6 \cdot \frac{3 \cdot 5 \cdot 8}{16^3} = \frac{6 \cdot 120}{4096} = \frac{720}{4096} = \frac{45}{256}$$

Proof II: With probability $3/16$ the first marble is blue and then with probability $5/16$ the second marble is red and then with probability $8/16$ the third marble is green. Hence, with probability

$$\frac{3}{16} \cdot \frac{5}{16} \cdot \frac{8}{16} = \frac{3 \cdot 5 \cdot 8}{16^3} = \frac{120}{4096} = \frac{15}{512}$$

the first marble is blue, the second marble is red, and the third marble is green. Similarly, with probability $15/512$ the color of the first marble is X, the color of the second marble is Y, and the color of the third marble is Z for any one of the 6 possible assignments of different colors out of blue, red and green to X, Y, and Z. Therefore, the probability of drawing 3 marbles of different colors is

$$6 \cdot \frac{15}{512} = \frac{90}{512} = \frac{45}{256}$$

v. What is the probability that $n \geq 2$ drawn marbles are of the same color?

Answer: $\frac{3^n+5^n+8^n}{16^n}$.

Proof I: There are 16^n ways to draw n marbles, there are 3^n ways to draw n blue marbles, there are 5^n ways to draw n red marbles, and there are 8^n ways to draw n green marbles. Therefore, the probability of drawing n marbles of the same color is

$$\frac{3^n + 5^n + 8^n}{16^n}$$

Proof II: With probability $(3/16)^n$ the n marbles are blue, with probability $(5/16)^n$ the n marbles are red, and with probability $(8/16)^n$ the n marbles are green. Therefore, the probability of drawing n marbles of the same color is

$$\left(\frac{3}{16}\right)^n + \left(\frac{5}{16}\right)^n + \left(\frac{8}{16}\right)^n = \frac{3^n}{16^n} + \frac{5^n}{16^n} + \frac{8^n}{16^n} = \frac{3^n + 5^n + 8^n}{16^n}$$

8. There are two bags of marbles. Each bag contains 6 marbles. The first contains 1 Red marble, 2 Blue marbles, and 3 Green marbles while the second contains 3 Red marbles, 2 Blue marbles, and 1 Green marble. You draw 1 random marble from each bag.

   Classifying all 36 combinations of one random marble from each bag: Denote Red by R, Blue by B, and Green by G. A pair XY for X and Y that could be either R or B or G means that an X marble was drawn from the first bag and a Y marble was drawn from the second bag. A straightforward counting implies that the $36 = 6 \times 6$ possible combinations can be partitioned into the following 9 categories:

3 RR combinations	2 RB combinations	1 RG combinations
6 BR combinations	4 BB combinations	2 BG combinations
9 GR combinations	6 GB combinations	3 GG combinations

   (a) What is the probability that both marbles are Red?

   Answer: $\frac{3}{36} = 0.0833\ldots \approx 8.33\%$.

   Proof: The event "both marbles are Red" happens only in the RR category. The probability of this event is therefore $3/36$.

   (b) What is the probability that at least one marble is Red?

   Answer: $\frac{21}{36} = 0.5833\ldots \approx 58.33\%$.

   Proof: The event "at least one marble is Red" happens in the RR, RB, RG, BR, and GR categories. There are $21 = 3 + 2 + 1 + 6 + 9$ combinations in these categories. Therefore, the probability of this event is $21/36$.

   (c) What is the probability that both marbles are Blue?

   Answer: $\frac{4}{36} = 0.1111\ldots \approx 11.11\%$.

   Proof: The event "both marbles are Blue" happens only in the BB category. The probability of this event is therefore $4/36$.

   (d) What is the probability that at least one marble is Blue?

   Answer: $\frac{20}{36} = 0.5555\ldots \approx 55.56\%$.

   Proof: The event "at least one marble is Blue" happens in the RB, BR, BB, BG, and GB categories. There are $20 = 2 + 6 + 4 + 2 + 6$ combinations in these categories. Therefore, the probability of this event is $20/36$.

   (e) What is the probability that both marbles are Green?

   Answer: $\frac{3}{36} = 0.0833\ldots \approx 8.33\%$.

   Proof: The event "both marbles are Green" happens only in the GG category. The probability of this event is therefore $3/36$.

   (f) What is the probability that at least one marble is Green?

   Answer: $\frac{21}{36} = 0.5833\ldots \approx 58.33\%$.

   Proof: The event "at least one marble is Green" happens in the RG, BG, GR, GB, and GG categories. There are $21 = 1 + 2 + 9 + 6 + 3$ combinations in these categories. Therefore, the probability of this event is $21/36$.

   Remark: Due to the symmetry between the Red and Green marbles the answers to parts (a) and (b) are identical to the answers to parts (e) and (f).

(g) What is the probability that both marbles have the same color?

Answer: $\frac{10}{36} = 0.2777\ldots \approx 27.78\%$.

Proof: The event "both marbles have the same color" happens in the RR, BB, and GG categories. There are $10 = 3 + 4 + 3$ combinations in these categories. Therefore, the probability of this event is $10/36$.

(h) What is the probability that both marbles have the same color given that none of the marbles is Red?

Answer: $\frac{7}{15} = 0.4666\ldots \approx 46.67\%$.

Proof: The event "both marbles have the same color none of the marbles is Red" happens in the BB and GG categories. There are $7 = 4 + 3$ combinations in these categories. The event "none of the marbles is Red" happens in the BB, BG, GB, and GG categories. There are $15 = 4+2+6+3$ combinations in these categories. Therefore, the probability for both marbles to have the same color given that none of the marbles is Red is $7/15$.

(i) What is the probability that both marbles have the same color given that none of the marbles is Blue?

Answer: $\frac{6}{16} = 0.375 = 37.5\%$.

Proof: The event "both marbles have the same color none of the marbles is Blue" happens in the RR and GG categories. There are $6 = 3 + 3$ combinations in these categories. The event "none of the marbles is Blue" happens in the RR, RG, GR, and GG categories. There are $16 = 3 + 1 + 9 + 3$ combinations in these categories. Therefore, the probability for both marbles to have the same color given that none of the marbles is Blue is $6/16$.

(j) What is the probability that both marbles have the same color given that none of the marbles is Green?

Answer: $\frac{7}{15} = 0.4666\ldots \approx 46.67\%$.

Proof: The event "both marbles have the same color none of the marbles is Green" happens in the RR and BB categories. There are $7 = 3 + 4$ combinations in these categories. The event "none of the marbles is Green" happens in the RR, RB, BR, and BB categories. There are $15 = 3 + 2 + 6 + 4$ combinations in these categories. Therefore, the probability for both marbles to have the same color given that none of the marbles is Green is $7/15$.

Remark: Due to the symmetry between the Red and Green marbles the answer to part (h) is identical to the answer to part (j).

(k) What is the probability that the two marbles have different colors?

Answer: $\frac{26}{36} = 0.7222\ldots \approx 72.22\%$.

Proof: The event "both marbles have different colors" happens in the RB, RG, BR, BG, GR, and GB categories. There are $26 = 2 + 1 + 6 + 2 + +9 + 6$ combinations in these categories. Therefore, the probability of this event is 26/36.

(l) What is the probability that the two marbles have different colors given that none of the marbles is Red?

Answer: $\frac{8}{15} = 0.5333\ldots \approx 53.33\%$.

Proof: The event "both marbles have different colors none of the marbles is Red" happens in the BG and GB categories. There are $8 = 2 + 6$ combinations in these categories. The event "none of the marbles is Red" happens in the BB, BG, GB, and GG categories. There are $15 = 4+2+6+3$ combinations in these categories. Therefore, the probability for both marbles to have different colors given that none of the marbles is Red is 8/15.

(m) What is the probability that the two marbles have different colors given that none of the marbles is Blue?

Answer: $\frac{10}{16} = 0.625 = 62.5\%$.

Proof: The event "both marbles have different colors none of the marbles is Blue" happens in the RG and GR categories. There are $10 = 1 + 9$ combinations in these categories. The event "none of the marbles is Blue" happens in the RR, RG, GR, and GG categories. There are $16 = 3 + 1 + 9 + 3$ combinations in these categories. Therefore, the probability for both marbles to have different colors given that none of the marbles is Blue is 10/16.

(n) What is the probability that the two marbles have different colors given that none of the marbles is Green?

Answer: $\frac{8}{15} = 0.5333\ldots \approx 53.33\%$.

Proof: The event "both marbles have different colors none of the marbles is Green" happens in the RB and BR categories. There are $8 = 2 + 6$ combinations in these categories. The event "none of the marbles is Green" happens in the RR, RB, BR, and BB categories. There are $15 = 3 + 2 + 6 + 4$ combinations in these categories. Therefore, the probability for both marbles to have different colors given that none of the marbles is Green is 8/15.

Remark 1: Due to the symmetry between the Red and Green marbles the answer to part (l) is identical to the answer to part (n).

Remark 2: Both marbles either have the same colors or different colors. As a result, the events in parts (k), (l), (m), and (n) are the complement events to those in parts (g), (h), (i), and (j) respectively. Consequently the sum of the answers in each of the following four pairs is 100%: part (g) and part (k), part (h) and part (l), part (i) and part (m), part (j) and part (n).

9. The six faces of a fair 6-face dice are numbered with $1, 2, 2, 3, 3, 3$ (instead of $1, 2, 3, 4, 5, 6$). As a result, a single throw of this dice shows 1 with probability $1/6$, shows 2 with probability $2/6 = 1/3$, and shows 3 with probability $3/6 = 1/2$. This dice is thrown twice.

   (a) What is the probability that the sum of both throws is 5?

   Answer: $\frac{1}{3} = 0.3333\ldots \approx 33.33\%$.

   Proof 1: There are two ways to get a sum of 5. Either the first throw shows 2 and the second throw shows 3 or the first throw shows 3 and the second throw shows 2. The first event happens with probability $1/3 \times 1/2$ and the second event happens with probability $1/2 \times 1/3$. Therefore, the probability that the sum of both throws is 5 is

$$\left(\frac{1}{3} \times \frac{1}{2}\right) + \left(\frac{1}{2} \times \frac{1}{3}\right) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

   Proof 2: The following table contains the sum of both throws in all possible 36 combinations

Sum	1	2	2	3	3	3
1	2	3	3	4	4	4
2	3	4	4	**5**	**5**	**5**
2	3	4	4	**5**	**5**	**5**
3	4	**5**	**5**	6	6	6
3	4	**5**	**5**	6	6	6
3	4	**5**	**5**	6	6	6

   The number 5 appears in the table 12 times. Therefore, the probability that the sum of both throws is 5 is $12/36 = 1/3$.

   (b) What is the probability that the sum of both throws is 5 given that none of them shows 1?

   Answer: $\frac{12}{25} = 0.48 = 48\%$.

   Proof 1: There are two ways to get a sum of 5 given that none of the throws shows 1. Either the first throw shows 2 and the second throw shows 3 or the first throw shows 3 and the second throw shows 2. Since it is given that none of the throws shows 1, it follows that the first event happens with probability $2/5 \times 3/5$ and the second event happens with probability $3/5 \times 2/5$. Therefore, the probability that the sum of both throws is 5 given that none of them shows 1 is

$$\left(\frac{2}{5} \times \frac{3}{5}\right) + \left(\frac{3}{5} \times \frac{2}{5}\right) = \frac{6}{25} + \frac{6}{25} = \frac{12}{25}$$

   Remark: This proof for part (b) follows the same arguments as those used in the first proof of part (a) with a fair 5-face dice whose faces are numbered $2, 2, 3, 3, 3$ instead of a fair 6-face dice whose faces are numbered $1, 2, 2, 3, 3, 3$.

   Proof 2: There are 25 entries in the table for the sum of both throws in which none of them shows 1. These are all the entries excluding the entries that appear in the first row and the first column. In 12 of these entries the sum is 5. Therefore, the probability that the sum of both throws is 5 given that none of them shows 1 is $12/25$.

(c) What is the probability that the product of both throws is 3?

Answer: $\frac{1}{6} = 0.1666\ldots \approx 16.67\%$.

Proof 1: There are two ways to get a product of 3. Either the first throw shows 1 and the second throw shows 3 or the first throw shows 3 and the second throw shows 1. The first event happens with probability $1/6 \times 1/2$ and the second event happens with probability $1/2 \times 1/6$. Therefore, the probability that the product of both throws is 3 is

$$\left(\frac{1}{6} \times \frac{1}{2}\right) + \left(\frac{1}{2} \times \frac{1}{6}\right) = \frac{1}{12} + \frac{1}{12} = \frac{1}{6}$$

Proof 2: The following table contains the product of both throws in all possible 36 combinations.

Sum	1	2	2	3	3	3
1	1	2	2	**3**	**3**	**3**
2	2	4	4	6	6	6
2	2	4	4	6	6	6
3	**3**	6	6	9	9	9
3	**3**	6	6	9	9	9
3	**3**	6	6	9	9	9

The number 3 appears in the table 6 times. Therefore, the probability that the product of both throws is 3 is $6/36 = 1/6$.

(d) What is the probability that the product of both throws is 3 given that none of them shows 2?

Answer: $\frac{3}{8} = 0.375 = 37.5\%$.

Proof: There are two ways to get a product of 3 given that none of them shows 2. Either the first throw shows 1 and the second throw shows 3 or the first throw shows 3 and the second throw shows 1. Since it is given that none of the throws shows 2, it follows that the first event happens with probability $1/4 \times 3/4$ and the second event happens with probability $3/4 \times 1/4$. Therefore, the probability that the product of both throws is 3 given that none of them shows 2 is

$$\frac{1}{4} \times \frac{3}{4} + \frac{3}{4} \times \frac{1}{4} = \frac{3}{16} + \frac{3}{16} = \frac{3}{8}$$

Remark: This proof for part (d) follows the same arguments as those used in the first proof of part (c) with a fair 4-face dice whose faces are numbered $1, 3, 3, 3$ instead of a fair 6-face dice whose faces are numbered $1, 2, 2, 3, 3, 3$.

Proof 2: There are 16 entries in the table for the product of both throws in which none of them shows 2. These are all the entries excluding the entries that appear in the first and second rows and the first and second columns. In 6 of these entries the product is 3. Therefore, the probability that the product of both throws is 3 given that none of the throws shows 2 is $6/16 = 3/8$.

(e) What is the probability that the sum of both throws is even?

Answer: $\frac{5}{9} = 0.5555\ldots \approx 55.56\%$.

Proof: The following table contains the sum of both throws in all possible 36 combinations.

Sum	1	2	2	3	3	3
1	**2**	3	3	**4**	**4**	**4**
2	3	**4**	**4**	5	5	5
2	3	**4**	**4**	5	5	5
3	**4**	5	5	**6**	**6**	**6**
3	**4**	5	5	**6**	**6**	**6**
3	**4**	5	5	**6**	**6**	**6**

Even numbers appear in the table 20 times. Therefore, the probability that the sum of both throws is even is $20/36 = 5/9$.

(f) What is the probability that the sum of both throws is even given that at least one throw shows 1?

Answer: $\frac{7}{11} = 0.6363\ldots \approx 63.64\%$.

Proof: Given that at least one throw shows 1, the only possible combinations in the table are the 11 entries that appear in the first row and the first column. 7 of these entries are even numbers. Therefore, the probability that the sum of both throws is even given that at least one throw shows 1 is $7/11$.

(g) What is the probability that the sum of both throws is even given that at least one throw shows 2?

Answer: $\frac{4}{20} = 0.2 = 20\%$.

Proof: Given that at least one throw shows 2, the only possible combinations in the table are the 20 entries that appear in the first and second rows and in the first and second columns. 4 of these entries are even numbers. Therefore, the probability that the sum of both throws is even given that at least one throw shows 2 is $4/20 = 1/5$.

(h) What is the probability that the sum of both throws is even given that at least one throw shows 3?

Answer: $\frac{5}{9} = 0.5555\ldots \approx 55.56\%$.

Proof: Given that at least one throw shows 3, the only possible combinations in the table are the 27 entries that appear in the first, second, and third rows and in the first, second, and third columns. 15 of these entries are even numbers. Therefore, the probability that the sum of both throws is even given that at least one throw shows 3 is $15/27 = 5/9$.

(i) What is the probability that the sum of both throws is odd?

Answer: $\frac{4}{9} = 0.4444\ldots \approx 44.44\%$.

Proof: The following table contains the sum of both throws in all possible 36 combinations.

Sum	1	2	2	3	3	3
1	2	**3**	**3**	4	4	4
2	**3**	4	4	**5**	**5**	**5**
2	**3**	4	4	**5**	**5**	**5**
3	4	**5**	**5**	6	6	6
3	4	**5**	**5**	6	6	6
3	4	**5**	**5**	6	6	6

Odd numbers appear in the table 16 times. Therefore, the probability that the sum of both throws is odd is $16/36 = 4/9$.

(j) What is the probability that the sum of both throws is odd given that at least one throw shows 1?

Answer: $\frac{4}{11} = 0.3636\ldots \approx 36.36\%$.

Proof: Given that at least one throw shows 1, the only possible combinations in the table are the 11 entries that appear in the first row and the first column. 4 of these entries are odd numbers. Therefore, the probability that the sum of both throws is odd given that at least one throw shows 1 is $4/11$.

(k) What is the probability that the sum of both throws is odd given that at least one throw shows 2?

Answer: $\frac{4}{5} = 0.8 = 80\%$.

Proof: Given that at least one throw shows 2, the only possible combinations in the table are the 20 entries that appear in the first and second rows and in the first and second columns. 16 of these entries are even numbers. Therefore, the probability that the sum of both throws is even given that at least one throw shows 2 is $16/20 = 4/5$.

(l) What is the probability that the sum of both throws is odd given that at least one throw shows 3?

Answer: $\frac{4}{9} = 0.4444\ldots \approx 44.44\%$.

Proof: Given that at least one throw shows 3, the only possible combinations in the table are the 27 entries that appear in the first, second, and third rows and in the first, second, and third columns. 12 of these entries are even numbers. Therefore, the probability that the sum of both throws is even given that at least one throw shows 3 is $12/27 = 4/9$.

Remark: The sum of both throws is either even or odd. As a result, the events in parts (i), (j), (k), and (l) are the complement events to those in parts (e), (f), (g), and (h) respectively. Consequently the sum of the answers in each of the following four pairs is 100%: part (e) and part (i), part (f) and part (j), part (g) and part (k), part (h) and part (l).