## Discrete Structures

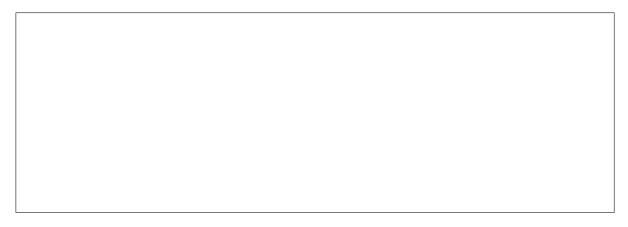
## Recursion Practice Problems

| Name: |  |  |  |
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| 1  | $\alpha$ 1 $\mu$ 1 | C 11 .        | recurrences      | 1   |       | 11 1 |            | 1          |     | 1        |
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$$T(n) = \begin{cases} 2 & \text{for } n = 1\\ T(n-1) + 7 & \text{for } n \ge 2 \end{cases}$$

$$T(n) = \begin{cases} 3 & \text{for } n = 1\\ 2T(n-1) & \text{for } n \ge 2 \end{cases}$$



$$T(n) = \begin{cases} 2 & \text{for } n = 1\\ (n+1)T(n-1) & \text{for } n \ge 2 \end{cases}$$



| 2. | Solve | the | following | recurrence | and | prove | that | vour | solution | is | correct. |
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$$P(n) = \begin{cases} 1 & \text{for } n = 0\\ 2 & \text{for } n = 1\\ 5P(n-1) - 6P(n-2) & \text{for } n \ge 2 \end{cases}$$

**Guide:** Do the bottom-up evaluation, guess the solution, and prove by induction the correctness of your guess.

| $F_n = \begin{cases} 0 & \text{for } n = 0\\ 1 & \text{for } n = 1\\ F_{n-1} + F_{n-2} & \text{for } n \ge 2 \end{cases}$                  |
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| What is the smallest n for which $F_n > 100$ ? What is the smallest n for which $F_n > 1000$ ?   |
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| Let $A_n = (F_1 + F_2 + \dots + F_n)/n$ be the average of the first $n$ Fibonacci numbers. What is the smallest $n$ for which $A_n > 10$ ? |
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| Find all $n$ for which $F_n = n$ . Explain why these are the only cases.   |
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| Find all n for which $F_n = n^2$ . Explain why these are the only cases.   |
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3. Some facts about the Fibonacci sequence:  $0,1,1,2,3,5,8,13,21,34,55,89,\ldots$ 

| bers and/or the Golden Ratio that do not appear in the class presentation. |  |  |  |  |  |  |
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| You do not need to provide proofs.   |  |  |  |  |  |  |
| Support your findings with pointers to their resources.                    |  |  |  |  |  |  |
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| 4. Prove the following identity for $n \geq$ | 4. | Prove | the | following | identity | for | $n \ge$ | 2 |
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$$F_{n+1} + F_{n-1} = F_{n+2} - F_{n-2}$$

**Hint:** There exists a simple proof without induction that is based on the recursive definition of the Fibonacci numbers.

| 5. | Define | the | following | (almost | Fibonacci | ) recurrence |
|----|--------|-----|-----------|---------|-----------|--------------|
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$$G_n = \begin{cases} 0 & \text{for } n = 0\\ 1 & \text{for } n = 1\\ G_{n-1} + G_{n-2} + 1 & \text{for } n \ge 2 \end{cases}$$

| $ \left(\begin{array}{c} G_{n-1} + G_{n-2} + 1 & \text{for } n \ge 2 \end{array}\right) $ |
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| Find the values of $G_0, G_1, \ldots, G_{10}$ .   |
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| Express $G_n$ as a function of Fibonacci numbers.   |
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| Prove that your expression for $G_n$ is correct for all $n \geq 0$ .                      |
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|  | $F_{2n+1} =$ | $F_{n+1}^2 + F_n^2$ |  |  |
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| 7. | For $n \geq 1$ , in how many out of the $n!$ permutations $\pi = (\pi(1), \pi(2), \dots, \pi(n))$ of the numbers $\{1, 2, \dots, n\}$ the value of $\pi(i)$ is either $i - 1$ , or $i$ , or $i + 1$ for all $1 \leq i \leq n$ ? |
|----|---|
|    | <b>Example:</b> The permutation (21354) follows the rules while the permutation (21534) does not because $\pi(3) = 5$ .   |
|    | <b>Hint:</b> Find the answer for small $n$ by checking all the permutations and then find the recursive formula depending on the possible values for $\pi(n)$ .   |
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