Discrete Structures

Recursion Practice Problems: Solutions

1. Solve the following recurrences and prove that your solutions are correct.

Recurrence I:

$$T(n) = \begin{cases} 2 & \text{for } n = 1 \\ T(n-1) + 7 & \text{for } n \geq 2 \end{cases}$$

Bottom-Up evaluation:

$$\begin{array}{rcl} T(1) & = & = 2 \ \ = 7 \cdot 1 - 5 \\ T(2) & = & T(1) + 7 = 9 \ \ = 7 \cdot 2 - 5 \\ T(3) & = & T(2) + 7 = 16 = 7 \cdot 3 - 5 \\ T(4) & = & T(3) + 7 = 23 = 7 \cdot 4 - 5 \\ & \vdots & \qquad\qquad\quad \vdots \\ T(n) & = & \qquad\qquad\quad = 7 \cdot n - 5 \end{array}$$

Top-Down evaluation:

$$\begin{array}{rcl} T(n) & = & T(n-1) + 7 \qquad\qquad\quad = T(n-1) + 1 \cdot 7 \\ & = & (T(n-2) + 7) + 1 \cdot 7 = T(n-2) + 2 \cdot 7 \\ & = & (T(n-3) + 7) + 2 \cdot 7 = T(n-3) + 3 \cdot 7 \\ & = & (T(n-4) + 7) + 3 \cdot 7 = T(n-4) + 4 \cdot 7 \\ & \vdots & \qquad\qquad\qquad\qquad \vdots \\ & = & \qquad\qquad\qquad\qquad = T(n-i) + i \cdot 7 \\ & \vdots & \qquad\qquad \iddots \\ & = & T(n-(n-1)) + (n-1)7 \\ & = & T(1) + 7n - 7 \\ & = & 2 + 7n - 7 \\ & = & 7n - 5 \end{array}$$

Solution: $\mathbf{T(n) = 7n - 5}$ for $n \geq 1$.

Proof by induction:

- *Induction base.* $T(1) = 2 = 7 \cdot 1 - 5$.
- *Induction hypothesis.* $T(n-1) = 7(n-1) - 5 = 7n - 12$ for $n > 1$.
- *Inductive step.* For $n > 1$,

$$\begin{array}{rcll} T(n) & = & T(n-1) + 7 & (* \text{ definition of } T(n) *) \\ & = & (7n - 12) + 7 & (* \text{ induction hypothesis } *) \\ & = & 7n - 5 & (* \text{ rearranging terms } *) \end{array}$$

Recurrence II:

$$T(n) = \begin{cases} 3 & \text{for } n = 1 \\ 2T(n-1) & \text{for } n \geq 2 \end{cases}$$

Bottom-Up evaluation:

$$\begin{array}{rcl} T(1) & = & = 3 = 3 \cdot 1 = 3 \cdot 2^0 \\ T(2) & = & 2T(1) = 6 = 3 \cdot 2 = 3 \cdot 2^1 \\ T(3) & = & 2T(2) = 12 = 3 \cdot 4 = 3 \cdot 2^2 \\ T(4) & = & 2T(3) = 24 = 3 \cdot 8 = 3 \cdot 2^3 \\ & \vdots & \qquad\qquad\qquad\qquad \vdots \\ T(n) & = & \qquad\qquad\qquad\qquad = 3 \cdot 2^{n-1} \end{array}$$

Top-Down evaluation:

$$\begin{array}{rcl} T(n) & = & 2 \cdot T(n-1) \qquad\qquad = 2^1 \cdot T(n-1) \\ & = & 2^1 \cdot (2 \cdot T(n-2)) = 2^2 \cdot T(n-2) \\ & = & 2^2 \cdot (2 \cdot T(n-3)) = 2^3 \cdot T(n-3) \\ & = & 2^3 \cdot (2 \cdot T(n-2)) = 2^4 \cdot T(n-4) \\ & \vdots & \qquad\qquad\qquad\quad \vdots \\ & = & \qquad\qquad\qquad\quad = 2^i \cdot T(n-i) \\ & \vdots & \qquad\qquad \iddots \\ & = & 2^{n-1} \cdot T(n-(n-1)) \\ & = & 2^{n-1} \cdot T(1) \\ & = & 2^{n-1} \cdot 3 \\ & = & 3 \cdot 2^{n-1} \end{array}$$

Solution: $\mathbf{T(n) = 3 \cdot 2^{n-1}}$ for $n \geq 1$.

Proof by induction:

- *Induction base.* $T(1) = 3 = 3 \cdot 1 = 3 \cdot 2^0 = 3 \cdot 2^{1-1}$.
- *Induction hypothesis.* $T(n-1) = 3 \cdot 2^{n-2}$ for $n > 1$.
- *Inductive step.* For $n > 1$,

$$\begin{array}{rcll} T(n) & = & 2 \cdot T(n-1) & (* \text{ definition of } T(n) *) \\ & = & 2 \cdot (3 \cdot 2^{n-2}) & (* \text{ induction hypothesis } *) \\ & = & 3 \cdot (2 \cdot 2^{n-2)} & (* \text{ rearranging terms } *) \\ & = & 3 \cdot 2^{n-1} & (* \text{ definition of the power function } *) \end{array}$$

Recurrence III:

$$T(n) = \begin{cases} 2 & \text{for } n = 1 \\ (n+1)T(n-1) & \text{for } n \geq 2 \end{cases}$$

Bottom-Up evaluation:

$$\begin{aligned}
T(1) &= & &= 2 = 2 \cdot 1 \\
T(2) &= & 3T(1) &= 6 = 3 \cdot 2 \cdot 1 \\
T(3) &= & 4T(2) &= 24 = 4 \cdot 3 \cdot 2 \cdot 1 \\
T(4) &= & 5T(3) &= 120 = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \\
&\vdots & & \quad \iddots \\
T(n) &= & (n+1) &\cdot n \cdot (n-1) \cdots 2 \cdot 1 \\
T(n) &= & (n+1)&!
\end{aligned}$$

Top-Down evaluation:

$$\begin{aligned}
T(n) &= & &= (n+1) \cdot T(n-1) \\
&= (n+1) \cdot (n \cdot T(n-2)) & &= (n+1) \cdot n \cdot T(n-2) \\
&= (n+1) \cdot n \cdot ((n-1) \cdot T(n-3)) & &= (n+1) \cdot n \cdot (n-1) \cdot T(n-3) \\
&= (n+1) \cdot n \cdot (n-1) \cdot ((n-2) \cdot T(n-4)) & &= (n+1) \cdot n \cdot (n-1) \cdot (n-2) \cdot T(n-4) \\
&\vdots & \iddots & \\
&= (n+1) \cdot n \cdot (n-1) \cdots 3 \cdot T(1) \\
&= (n+1) \cdot n \cdot (n-1) \cdots 3 \cdot 2 \\
&= (n+1) \cdot n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1 \\
&= (n+1)!
\end{aligned}$$

Solution: $\mathbf{T(n) = (n+1)!}$ for $n \geq 1$.

Proof by induction:

- *Induction base.* $T(1) = 2 = 2 \cdot 1 = 2! = (1+1)!$.
- *Induction hypothesis.* $T(n-1) = n!$ for $n > 1$.
- *Inductive step.* For $n \geq 1$,

$$\begin{aligned}
T(n) &= (n+1)T(n-1) & &(* \text{ definition of } T(n) *) \\
&= (n+1)n! & &(* \text{ induction hypothesis } *) \\
&= (n+1)! & &(* \text{ definition of factorial } *)
\end{aligned}$$

2. Solve the following recurrence and prove that your solution is correct.

$$P(n) = \begin{cases} 1 & \text{for } n = 0 \\ 2 & \text{for } n = 1 \\ 5P(n-1) - 6P(n-2) & \text{for } n \geq 2 \end{cases}$$

Small values of n:

$$\begin{array}{lclclcr} P(0) & = & & & & = & 1 \\ P(1) & = & & & & = & 2 \\ P(2) & = & 5 \cdot 2 - 6 \cdot 1 & = & 10 - 6 & = & 4 \\ P(3) & = & 5 \cdot 4 - 6 \cdot 2 & = & 20 - 12 & = & 8 \\ P(4) & = & 5 \cdot 8 - 6 \cdot 4 & = & 40 - 24 & = & 16 \\ P(5) & = & 5 \cdot 16 - 6 \cdot 8 & = & 80 - 48 & = & 32 \end{array}$$

Solution: $\mathbf{P(n) = 2^n}$ for $n \geq 0$.

Proof by induction:

- *Induction base.* $P(0) = 1 = 2^0$ and $P(1) = 2 = 2^1$.
- *Induction hypothesis.* $P(n-1) = 2^{n-1}$ and $P(n-2) = 2^{n-2}$ for $n \geq 2$.
- *Inductive step.* For $n \geq 2$,

$$\begin{array}{lcll} P(n) & = & 5P(n-1) - 6P(n-2) & (* \text{ definition of } P(n) *) \\ & = & 5 \cdot 2^{n-1} - 6 \cdot 2^{n-2} & (* \text{ induction hypothesis } *) \\ & = & 5 \cdot 2^{n-1} - 3 \cdot 2^{n-1} & (* \text{ algebra } *) \\ & = & 2 \cdot 2^{n-1} & (* \text{ algebra } *) \\ & = & 2^n & (* \text{ algebra } *) \end{array}$$

3. Some facts about the Fibonacci sequence: $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$

$$F_n = \begin{cases} 0 & \text{for } n = 0 \\ 1 & \text{for } n = 1 \\ F_{n-1} + F_{n-2} & \text{for } n \geq 2 \end{cases}$$

- What is the smallest n for which $F_n > 100$?

  Answer: $F_{11} = 89$ and $F_{12} = 144$, therefore 12 is the smallest n for which $F_n > 100$.

- What is the smallest n for which $F_n > 1000$?

  Answer: $F_{16} = 987$ and $F_{17} = 1597$, therefore 17 is the smallest n for which $F_n > 1000$.

- Let $A_n = (F_1 + F_2 + \cdots + F_n)/n$ be the average of the first n Fibonacci numbers. What is the smallest n for which $A_n > 10$?

  Answer: Recall that $\sum_{i=1}^{n} F_i = F_{n+2} - 1$. Therefore the sequence $A_1, A_2, \ldots$ is

n	1	2	3	4	5	6	7	8	9	10
$A(n)$	1	1	4/3	7/4	12/5	20/6	33/7	54/8	88/9	143/10

  Since $A_9 = 88/9 < 10$ and $A_{10} = 143/10 > 10$, it follows that 10 is the smallest n for which $A_n > 10$.

- Find all n for which $F_n = n$.

  Answer: By inspection, $F_0 = 0$, $F_1 = 1$, and $F_5 = 5$ while $F_2 \neq 2$, $F_3 \neq 3$, and $F_4 \neq 4$. Since F_n as a function of n grows faster than the function n for $n > 5$ (see the remark below), it follows that $F_n > n$ for $n > 5$.

- Find all n for which $F_n = n^2$.

  Answer: By inspection, $F_0 = 0 = 0^2$, $F_1 = 1 = 1^2$, and $F_{12} = 144 = 12^2$ while $F_n \neq n^2$ for $n \in \{2, 3, \ldots, 11\}$. Since F_n as a function of n grows faster than the function n^2 for $n > 12$ (see the remark below), it follows that $F_n > n^2$ for $n > 12$.

Remark: $F_{n+1}/F_n \approx \phi = 1.618\ldots$. Therefore F_n as a function of n grows faster than the function n^2 for which $(n+1)^2/n^2$ approaches 1 as n tends to infinity. In particular after $F_{12} = 144$ for which $F_n = n^2$, it is always the case that $F_n > n^2$ for $n > 12$. This can be proven by induction. Similarly, after $F_5 = 5$ for which $F_n = n$, it is always the case that $F_n > n$ for $n > 5$.

4. Prove the following identity for $n \geq 2$:

$$F_{n+1} + F_{n-1} = F_{n+2} - F_{n-2}$$

The cases $n = 2, 3, 4, 5, 6$**:**

$$\begin{array}{ccccccccc}
F_3 + F_1 & = & 2+1 & = & 3 & = & 3-0 & = & F_4 - F_0 \\
F_4 + F_2 & = & 3+1 & = & 4 & = & 5-1 & = & F_5 - F_1 \\
F_5 + F_3 & = & 5+2 & = & 7 & = & 8-1 & = & F_6 - F_2 \\
F_6 + F_4 & = & 8+3 & = & 11 & = & 13-2 & = & F_7 - F_3 \\
F_7 + F_5 & = & 13+5 & = & 18 & = & 21-3 & = & F_8 - F_4
\end{array}$$

Proof: For $n \geq 2$,

$$\begin{array}{rcll}
F_{n+1} + F_{n-1} & = & F_{n+1} + (F_n - F_{n-2}) & (*\ F_{n-1} = F_n - F_{n-2}\ *) \\
 & = & (F_{n+1} + F_n) - F_{n-2} & (*\ \textit{rearranging parenthesis}\ *) \\
 & = & F_{n+2} - F_{n-2} & (*\ F_{n+2} = F_{n+1} + F_n\ *)
\end{array}$$

5. Define the following (*almost Fibonacci*) recurrence

$$G_n = \begin{cases} 0 & \text{for } n = 0 \\ 1 & \text{for } n = 1 \\ G_{n-1} + G_{n-2} + 1 & \text{for } n \geq 2 \end{cases}$$

The first 11 values in the sequence $G_0, G_1, \ldots, G_{10}$ are:

n	0	1	2	3	4	5	6	7	8	9	10
G_n	0	1	2	4	7	12	20	33	54	88	143

There are two "natural" guesses for G_n as a function of Fibonacchi numbers based on the first 13 numbers in the Fibonacci sequence:

n	0	1	2	3	4	5	6	7	8	9	10	11	12
F_n	0	1	1	2	3	5	8	13	21	34	55	89	144

$$\begin{aligned} G_n &= F_{n+2} - 1 \\ G_n &= F_0 + F_1 + \cdots + F_n \end{aligned}$$

Remark: Recall that $\mathbf{F_0 + F_1 + \cdots F_n = F_{n+2} - 1}$ is an identity that can be proven by induction.

Proposition 1: $G_n = F_{n+2} - 1$ for $n \geq 0$.

Proof by induction:

- *Induction base.* Show that $G_0 = F_2 - 1$ and that $G_1 = F_3 - 1$:

$$\begin{aligned} G_0 &= 0 = 1 - 0 = F_2 - 1 \\ G_1 &= 1 = 2 - 1 = F_3 - 1 \end{aligned}$$

- *Induction hypothesis.* Assume that for $n \geq 2$:

$$\begin{aligned} G_{n-1} &= F_{n+1} - 1 \\ G_{n-2} &= F_n - 1 \end{aligned}$$

- *Inductive step.* Prove that $G_n = F_{n+2} - 1$ for $n \geq 2$:

$$\begin{aligned} G_n &= G_{n-1} + G_{n-2} + 1 && (* \text{ definition of } G_n *) \\ &= (F_{n+1} - 1) + (F_n - 1) + 1 && (* \text{ induction hypothesis } *) \\ &= (F_{n+1} + F_n) - 1 && (* \text{ simplifying } *) \\ &= F_{n+2} - 1 && (* \ F_{n+2} = F_{n+1} + F_n *) \end{aligned}$$

Proposition 2: $G_n = F_0 + F_1 + \cdots + F_n$ for $n \geq 0$.

Proof by induction:

- *Induction base.* Show that $G_0 = F_0$ and that $G_1 = F_0 + F_1$:

$$\begin{aligned} G_0 &= 0 = F_0 \\ G_1 &= 1 = 0 + 1 = F_0 + F_1 \end{aligned}$$

- *Induction hypothesis.* Assume that for $n \geq 2$:

$$\begin{aligned} G_{n-1} &= F_0 + F_1 + F_2 + \cdots + F_{n-1} \\ G_{n-2} &= F_0 + F_1 + \cdots + F_{n-2} \end{aligned}$$

- *Inductive step.* Prove that $G_n = F_0 + F_1 + \cdots + F_n$ for $n \geq 2$:

$$\begin{aligned} G_n &= G_{n-1} + G_{n-2} + 1 && \text{(* definition of } G_n \text{ *)} \\ &= (F_0 + F_1 + F_2 + \cdots + F_{n-1}) + (F_0 + F_1 + \cdots + F_{n-2}) + 1 && \text{(* induction hypothesis *)} \\ &= (F_0 + 1) + (F_1 + F_2 + \cdots + F_{n-1}) + (F_0 + F_1 + \cdots + F_{n-2}) && \text{(* rearranging terms *)} \\ &= F_1 + (F_1 + F_2 + \cdots + F_{n-1}) + (F_0 + F_1 + \cdots + F_{n-2}) && \text{(* } F_0 + 1 = F_1 \text{ *)} \\ &= F_1 + (F_1 + F_0) + (F_2 + F_1) + \cdots + (F_{n-1} + F_{n-2}) && \text{(* rearranging terms *)} \\ &= F_1 + F_2 + F_3 + \cdots + F_n && \text{(* Fibonacci recurrence *)} \\ &= F_0 + F_1 + F_2 + F_3 + \cdots + F_n && \text{(* F}_0 = 0 \text{ *)} \end{aligned}$$

6. Prove the following identity for $n \geq 1$:

$$F_{2n+1} = F_{n+1}^2 + F_n^2$$

Proof by induction:

- *Induction base.* Verify correctness for $n = 1, 2, 3, 4$:

$$\begin{array}{lllllll} F_3 & = & 2 & = & 1^2 + 1^2 & = & F_2^2 + F_1^2 \\ F_5 & = & 5 & = & 2^2 + 1^2 & = & F_3^2 + F_2^2 \\ F_7 & = & 13 & = & 3^2 + 2^2 & = & F_4^2 + F_3^2 \\ F_9 & = & 34 & = & 5^2 + 3^2 & = & F_5^2 + F_4^2 \end{array}$$

- *Induction hypothesis.* Assume that for $n \geq 3$:

$$\begin{array}{lll} F_{2n-1} & = & F_n^2 + F_{n-1}^2 \\ F_{2n-3} & = & F_{n-1}^2 + F_{n-2}^2 \end{array}$$

- *Inductive step.* By replacing F_{2n+1} with $F_{2n} + F_{2n-1}$, then replacing F_{2n} with $F_{2n-1} + F_{2n-2}$ and combining terms, and then replacing F_{2n-2} with $F_{2n-1} - F_{2n-3}$ and combining terms, F_{2n+1} becomes a function of F_{2n-1} and F_{2n-3}.

$$\begin{array}{lll} F_{2n+1} & = & F_{2n} + F_{2n-1} \\ & = & (F_{2n-1} + F_{2n-2}) + F_{2n-1} \\ & = & 2F_{2n-1} + F_{2n-2} \\ & = & 2F_{2n-1} + (F_{2n-1} - F_{2n-3}) \\ & = & 3F_{2n-1} - F_{2n-3} \end{array}$$

  After applying the induction hypothesis for F_{2n-1} and F_{2n-3} and simplifying, F_{2n+1} becomes a function of the squares of F_n, F_{n-1}, and F_{n-2}.

$$\begin{array}{lll} F_{2n+1} & = & 3F_{2n-1} - F_{2n-3} \\ & = & 3(F_n^2 + F_{n-1}^2) - (F_{n-1}^2 + F_{n-2}^2) \\ & = & 3F_n^2 + 2F_{n-1}^2 - F_{n-2}^2 \end{array}$$

  By replacing F_{n-2} with $F_n - F_{n-1}$ and simplifying, F_{2n+1} becomes a function of the squares of F_n and F_{n-1}.

$$\begin{array}{lll} F_{2n+1} & = & 3F_n^2 + 2F_{n-1}^2 - F_{n-2}^2 \\ & = & 3F_n^2 + 2F_{n-1}^2 - (F_n - F_{n-1})^2 \\ & = & 3F_n^2 + 2F_{n-1}^2 - (F_n^2 - 2F_{n-1}F_n + F_{n-1}^2) \\ & = & 2F_n^2 + F_{n-1}^2 + 2F_{n-1}F_n \\ & = & (F_n^2 + 2F_{n-1}F_n + F_{n-1}^2) + F_n^2 \\ & = & (F_n + F_{n-1})^2 + F_n^2 \end{array}$$

  The proof is completed by replacing $F_n + F_{n-1}$ with F_{n+1}.

$$\begin{array}{lll} F_{2n+1} & = & (F_n + F_{n-1})^2 + F_n^2 \\ & = & F_{n+1}^2 + F_n^2 \end{array}$$

7. For $n \geq 1$, there are F_{n+1} permutations $\pi = (\pi(1), \pi(2), \ldots, \pi(n))$ of the numbers $\{1, 2, \ldots, n\}$ in which the value of $\pi(i)$ is either $i-1$, or i, or $i+1$ for all $1 \leq i \leq n$ where F_n is the n^{th} Fibonacci number.

Proof: For $n \geq 1$, a permutation π is ***good*** if $\pi(i)$ is either $i-1$, or i, or $i+1$ for all $1 \leq i \leq n$. Otherwise, π is a ***bad*** permutation. Equivalently, if π is a ***bad*** permutation then there exists at least one index i for which $|\pi(i) - i| \geq 2$ while if π is a ***good*** permutation then $|\pi(i) - i| \leq 1$ for all $1 \leq i \leq n$.

$n = 1$**:** The only permutation (1) is a ***good*** permutation.

$n = 2$**:** The two permutations $(1, 2)$ and $(2, 1)$ are ***good*** permutations.

$n = 3$**:** Out of the six permutations, the three permutations $(1, 2, 3)$, $(2, 1, 3)$, and $(1, 3, 2)$ are ***good*** permutations while the three permutations $(3, 1, 2)$, $(2, 3, 1)$, and $(3, 2, 1)$ are ***bad*** permutations since either $\pi(1) = 3$ or $\pi(3) = 1$.

$n = 4$**:** Out of the 24 permutations, only the five permutations $(1, 2, 3, 4)$, $(2, 1, 3, 4)$, $(1, 3, 2, 4)$, $(1, 2, 4, 3)$, and $(2, 1, 4, 3)$ are ***good*** permutations.

For $n \geq 1$, let G_n be the number of ***good*** permutations. The above shows that $G_1 = 1 = F_2$, $G_2 = 2 = F_3$, $G_3 = 3 = F_4$, and $G_4 = 5 = F_5$.

For $n \geq 3$, let π be a ***good*** permutation. Therefore, $\pi(n) = n$ or $\pi(n) = n - 1$ because otherwise $|\pi(n) - n| \geq 2$.

- Assume $\pi(n) = n$. Then $\pi' = (\pi(1), \pi(2), \ldots, \pi(n-1))$ must be a ***good*** permutation for the numbers $\{1, 2, \ldots, n-1\}$. It follows that there are G_{n-1} ***good*** permutations π in which $\pi(n) = n$.
- Assume $\pi(n) = n-1$. Then $\pi(n-1) = n$ because otherwise $\pi(i) = n$ for some $i \leq n-2$ which contradicts the ***goodness*** of π. Moreover, $\pi'' = (\pi(1), \pi(2), \ldots, \pi(n-2))$ must be a ***good*** permutation for the numbers $\{1, 2, \ldots, n-2\}$. It follows that there are G_{n-2} ***good*** permutations π in which $\pi(n) = n-1$ and $\pi(n-1) = n$.

The above two cases imply that $G_n = G_{n-1} + G_{n-2}$ for $n \geq 3$. Since $G_1 = 1 = F_2$ and $G_2 = 2 = F_3$, it follows that the number of ***good*** permutations for $n \geq 1$ is the Fibonacci number F_{n+1}.

Example: For $n = 5$, there are $F_6 = 8$ ***good*** permutations. In $F_5 = 5$ of them $\pi(5) = 5$:

$$(1, 2, 3, 4, 5) \quad (2, 1, 3, 4, 5) \quad (1, 3, 2, 4, 5) \quad (1, 2, 4, 3, 5) \quad (2, 1, 4, 3, 5)$$

and in $F_4 = 3$ of them $\pi(4) = 5$ and $\pi(5) = 4$:

$$(1, 2, 3, 5, 4) \quad (2, 1, 3, 5, 4) \quad (1, 3, 2, 5, 4)$$