Discrete Math

Sets Practice Problems: Solutions

1. Let U be the set of all the positive integers smaller than 19, let A be the set of all the even positive integers smaller than 19, and let B be the set of all the positive integers smaller than 19 that are divided by 3. The complement of sets are defined relative to the set U which contains both A and B.

$$\begin{aligned}
U &= \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18\} \\
A &= \{2,4,6,8,10,12,14,16,18\} \\
B &= \{3,6,9,12,15,18\} \\
\neg A &= \{1,3,5,7,9,11,13,15,17\} \\
\neg B &= \{1,2,4,5,7,8,10,11,13,14,16,17\} \\
A \cup B &= \{2,3,4,6,8,9,10,12,14,15,16,18\} \\
A \cap B &= \{6,12,18\} \\
A \setminus B &= \{2,4,8,10,14,16\} \\
B \setminus A &= \{3,9,15\} \\
A \triangle B &= \{2,3,4,8,9,10,14,15,16\} \\
\neg(A \cup B) &= \{1,5,7,11,13,17\} \\
\neg(A \cap B) &= \{1,2,3,4,5,7,8,9,10,11,13,14,15,16,17\}
\end{aligned}$$

2. The following are four ways to write the two De Morgan's laws for the four sets A, B, C, and D using the four negation notations for a set X: X', $\overline{X}$, $\neg X$, and X^C.

$$\begin{aligned}
(A \cup B \cup C \cup D)' &= A' \cap B' \cap C' \cap D' & (A \cap B \cap C \cap D)' &= A' \cup B' \cup C' \cup D' \\
\overline{A \cup B \cup C \cup D} &= \overline{A} \cap \overline{B} \cap \overline{C} \cap \overline{D} & \overline{A \cap B \cap C \cap D} &= \overline{A} \cup \overline{B} \cup \overline{C} \cup \overline{D} \\
\neg(A \cup B \cup C \cup D) &= \neg A \cap \neg B \cap \neg C \cap \neg D & \neg(A \cap B \cap C \cap D) &= \neg A \cup \neg B \cup \neg C \cup \neg D \\
(A \cup B \cup C \cup D)^C &= A^C \cap B^C \cap C^C \cap D^C & (A \cap B \cap C \cap D)^C &= A^C \cup B^C \cup C^C \cup D^C
\end{aligned}$$

3. The following is the principle of inclusion exclusion for the five sets A, B, C, D, and E.

$$\begin{aligned}
|A \cup B \cup C \cup D \cup E| = \; & +|A| + |B| + |C| + |D| + |E| \\
& -|A \cap B| - |A \cap C| - |A \cap D| - |A \cap E| - |B \cap C| \\
& -|B \cap D| - |B \cap E| - |C \cap D| - |C \cap E| - |D \cap E| \\
& +|A \cap B \cap C| + |A \cap B \cap D| + |A \cap B \cap E| + |A \cap C \cap D| + |A \cap C \cap E| \\
& +|A \cap D \cap E| + |B \cap C \cap D| + |B \cap C \cap E| + |B \cap D \cap E| + |C \cap D \cap E| \\
& -|A \cap B \cap C \cap D| - |A \cap B \cap C \cap E| - |A \cap B \cap D \cap E| \\
& -|A \cap C \cap D \cap E| - |B \cap C \cap D \cap E| \\
& +|A \cap B \cap C \cap D \cap E|
\end{aligned}$$

4. Prove that the distributive laws for sets are correct.

(a) $(A \cup (B \cap C)) = ((A \cup B) \cap (A \cup C))$

Proof:

- Let $x \in (A \cup (B \cap C))$. There are two cases. In the first case $x \in A$, and therefore $x \in (A \cup B)$ and $x \in (A \cup C)$, which imply that $x \in ((A \cup B) \cap (A \cup C))$. In the second case $x \in (B \cap C)$, and therefore $x \in B$ and $x \in C$, which imply that $x \in (A \cup B)$ and $x \in (A \cup C)$, which imply that $x \in ((A \cup B) \cap (A \cup C))$. Since this holds for any $x \in (A \cup (B \cap C))$, it follows that

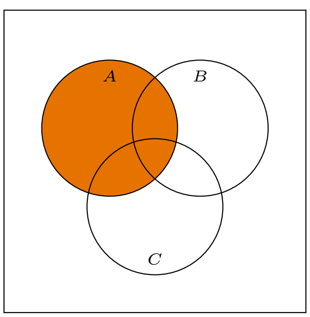
$$(A \cup (B \cap C)) \subseteq ((A \cup B) \cap (A \cup C))$$

- Let $x \in ((A \cup B) \cap (A \cup C))$. Therefore, $x \in (A \cup B)$ and $x \in (A \cup C)$. There are two cases. In the first case $x \in A$, and therefore $x \in (A \cup (B \cap C))$. In the second case $x \notin A$, and therefore $x \in B$ and $x \in C$, which imply that $x \in (B \cap C)$, which implies that $x \in (A \cup (B \cap C))$. Since this holds for any $x \in ((A \cup B) \cap (A \cup C))$, it follows that

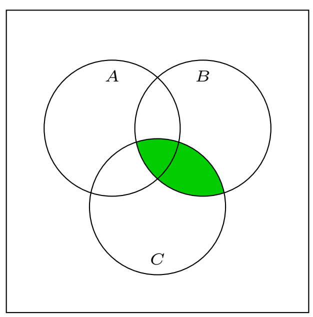
$$((A \cup B) \cap (A \cup C)) \subseteq (A \cup (B \cap C))$$

- For two sets X and Y if both $X \subseteq Y$ and $Y \subseteq X$ hold then $X = Y$. Therefore,

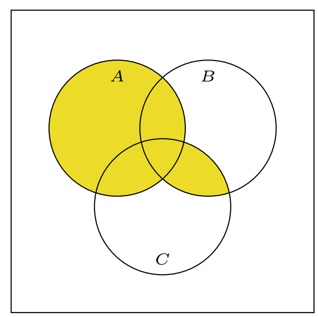
$$(A \cup (B \cap C)) = ((A \cup B) \cap (A \cup C))$$



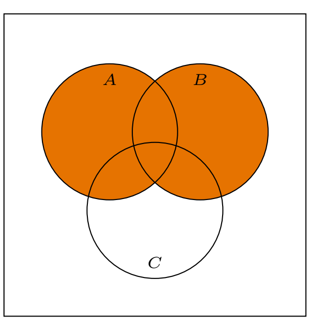
A



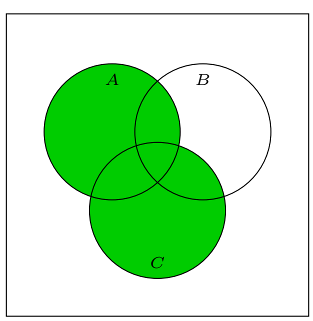
$B \cap C$



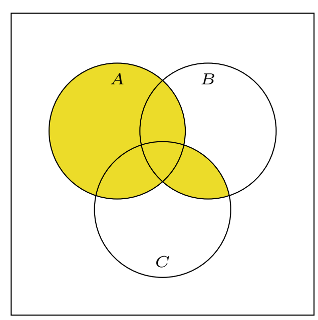
$A \cup (B \cap C)$



$A \cup B$



$A \cup C$



$(A \cup B) \cap (A \cup C)$

(b) $(A \cap (B \cup C)) = ((A \cap B) \cup (A \cap C))$

Proof:

- Let $x \in (A \cap (B \cup C))$. Therefore $x \in A$. There are two cases. In the first case $x \in B$, and therefore $x \in (A \cap B)$, which implies that $x \in ((A \cap B) \cup (A \cap C))$. In the second case $x \in C$, and therefore $x \in (A \cap C)$, which implies that $x \in ((A \cap B) \cup (A \cap C))$. Since this holds for any $x \in (A \cap (B \cup C))$, it follows that

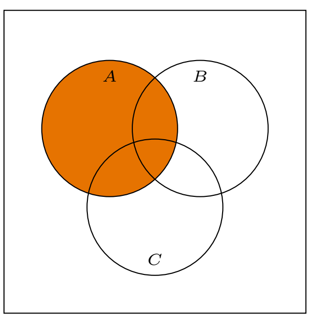
$$(A \cap (B \cup C)) \subseteq ((A \cap B) \cup (A \cap C))$$

- Let $x \in ((A \cap B) \cup (A \cap C))$. There are two cases. In the first case $x \in (A \cap B)$, which implies that $x \in A$ and $x \in (B \cup C)$, and therefore $x \in (A \cap (B \cup C))$. In the second case $x \in (A \cap C)$, which implies that $x \in A$ and $x \in (B \cup C)$, and therefore $x \in (A \cap (B \cup C))$. Since this holds for any $x \in ((A \cap B) \cup (A \cap C))$, it follows that

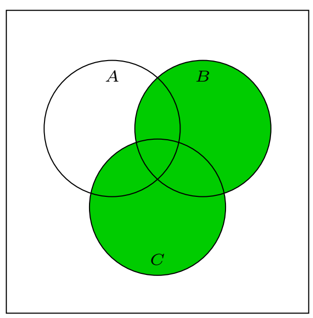
$$((A \cap B) \cup (A \cap C)) \subseteq (A \cap (B \cup C))$$

- For two sets X and Y if both $X \subseteq Y$ and $Y \subseteq X$ then $X = Y$. Therefore,

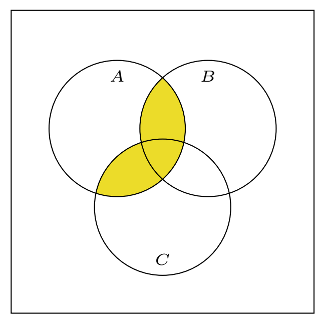
$$(A \cap (B \cup C)) = ((A \cap B) \cup (A \cap C))$$



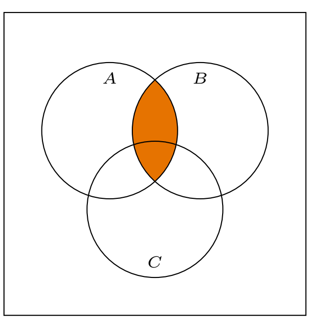
A



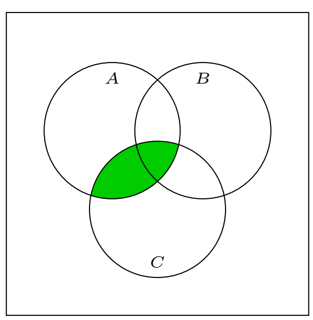
$B \cup C$



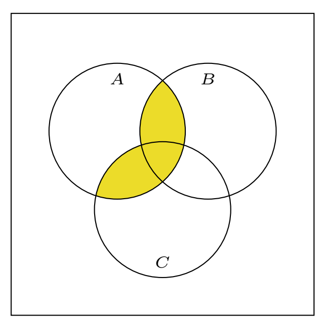
$A \cap (B \cup C)$



$A \cap B$



$A \cap C$



$(A \cap B) \cup (A \cap C)$

5. Prove the following two statements:

   (a) If $A \subseteq B$ and $A \subseteq C$ then $A \subseteq (B \cap C)$

   Proof: Let $x \in A$. Since $A \subseteq B$, it follows that $x \in B$ and since $A \subseteq C$, it follows that $x \in C$. Since $x \in B$ and $x \in C$, it follows that $x \in (B \cap C)$. Since this holds for any $x \in A$, it follows that $A \subseteq (B \cap C)$.

   (b) If $A \subseteq C$ and $B \subseteq C$ then $(A \cup B) \subseteq C$

   Proof: Let $x \in (A \cup B)$. If $x \in A$ then $x \in C$ since $A \subseteq C$ and if $x \in B$ then $x \in C$ since $B \subseteq C$. Since this holds for any $x \in (A \cup B)$, it follows that $(A \cup B) \subseteq C$.

6. Prove or give a counterexample for each of the following two statements:

   (a) If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$

   Proof: Let $x \in A$. Then $x \in B$ since $A \subseteq B$ and $x \in C$ since $B \subseteq C$. Since this holds for any $x \in A$, it follows that $A \subseteq C$.

   (b) If $A \in B$ and $B \in C$ then $A \in C$

   Counterexample:

   $$A = \{1\} \qquad B = \{\{1\}\} \qquad C = \{\{\{1\}\}\}$$

   It follows that $A \in B$ and $B \in C$ but $A \notin C$.

7. Let A and B be two sets. If $2^A \subseteq 2^B$, what is the relation between A and B? (For a set X, the set 2^X is the set of all the subsets of X.)

   Answer: $A \subseteq B$.

   Proof 1:

   - Since $A \subseteq A$, it follows that $A \in 2^A$.
   - Since $2^A \subseteq 2^B$, it follows that $A \in 2^B$.
   - Since $A \in 2^B$, it follows that $A \subseteq B$.

   Proof 2:

   - Let $x \in A$.
   - Since $\{x\} \subseteq A$, it follows that $\{x\} \in 2^A$.
   - Since $2^A \subseteq 2^B$, it follows that $\{x\} \in 2^B$.
   - Since $\{x\} \in 2^B$, it follows that $\{x\} \subseteq B$.
   - Since $\{x\} \subseteq B$, it follows that $x \in B$.
   - $A \subseteq B$ because this holds for any $x \in A$.

8. Prove or disprove: for any three sets A, B, and C: $C \setminus (A \cap B) = (C \setminus A) \cap (C \setminus B)$.

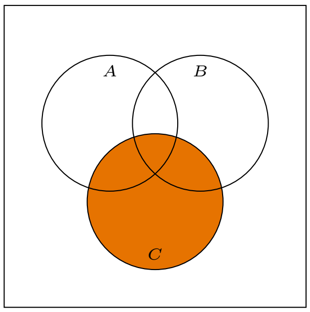
Counterexample: Define $A = \{1, 2\}$, $B = \{1, 3\}$, and $C = \{1, 2, 3, 4\}$.

(a) $A \cap B = \{1\}$ and therefore $C \setminus (A \cap B) = \{2, 3, 4\}$.

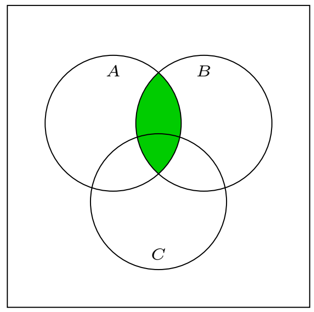
(b) $C \setminus A = \{3, 4\}$ and $C \setminus B = \{2, 4\}$ and therefore $(C \setminus A) \cap (C \setminus B) = \{4\}$.

(a) and (b) imply that for these three sets A, B, and C

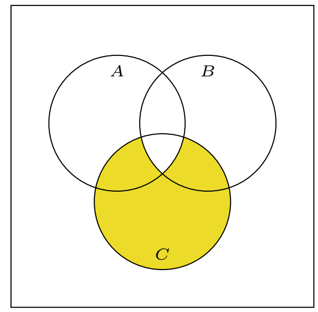
$$C \setminus (A \cap B) \neq (C \setminus A) \cap (C \setminus B)$$



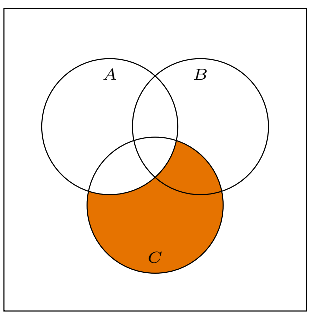
C



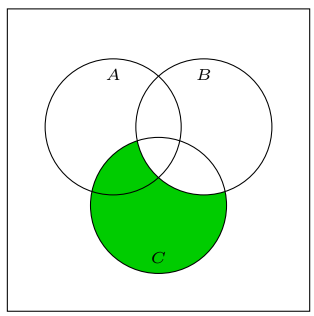
$A \cap B$



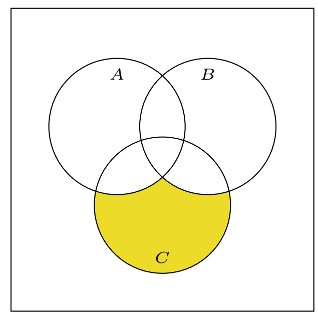
$C \setminus (A \cap B)$



$C \setminus A$



$C \setminus B$

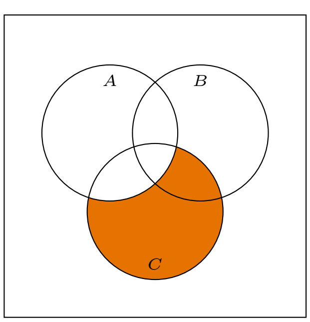


$(C \setminus A) \cap (C \setminus B)$

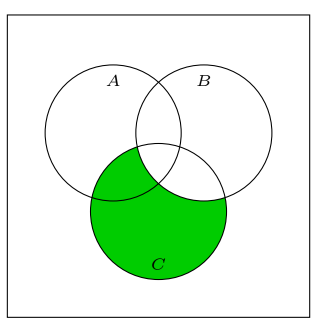
Remark 1: Observe that $((C \setminus A) \cap (C \setminus B)) \subseteq (C \setminus (A \cap B))$. The inequality happens when either $(C \cap A) \setminus B$ is not empty, or $(C \cap B) \setminus A$ is not empty, or both are not empty. In the counterexample, $(C \cap A) \setminus B = \{2\}$ and $(C \cap B) \setminus A = \{3\}$.

Remark 2: The correct equality is the following version of one of the De Morgan's laws:

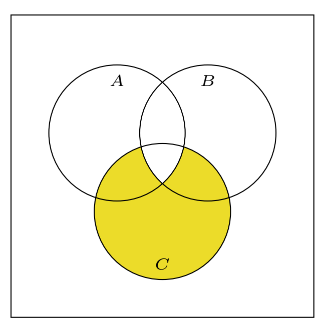
$$C \setminus (A \cap B) = (C \setminus A) \cup (C \setminus B)$$



$C \setminus A$



$C \setminus B$



$(C \setminus A) \cup (C \setminus B)$

9. Let $a(t) \le b(t) \le c(t)$ be the lengths of the three sides of a triangle t in a non-decreasing order. Define the following sets:

   - T: The set of all triangles.
   - X: The set of all triangles t for which $a(t) = b(t)$.
   - Y: The set of all triangles t for which $b(t) = c(t)$.

   Using only set operations on these three set, define the following sets.

   (a) $X \cap Y$ is the set of all equilateral triangles (all sides equal).

   Explanation: Let E be the set of all equilateral triangles.

   - If $t \in E$ then $a(t) = b(t) = c(t)$. Therefore, $t \in X$ and $t \in Y$, which imply that $t \in X \cap Y$. Consequently, $E \subseteq (X \cap Y)$.
   - If $t \in X \cap Y$ then $a(t) = b(t)$ and $b(t) = c(t)$ and therefore $a(t) = b(t) = c(t)$, which implies that $t \in E$. Consequently, $(X \cap Y) \subseteq E$.
   - For two sets A and B, if both $A \subseteq B$ and $B \subseteq A$ then $A = B$. Therefore, $E = X \cap Y$.

   (b) $X \cup Y$ is the set of all isosceles triangles (at least two sides equal).

   Explanation: Let I be the set of all isosceles triangles.

   - If $t \in I$ then either $a(t) = b(t)$, or $b(t) = c(t)$, or $a(t) = c(t)$. In the third case, since $a(t) \le b(t) \le c(t)$, it follows that $a(t) = b(t) = c(t)$. Therefore, in the first case, $t \in X$, in the second case $t \in Y$, and in the third case $t \in (X \cap Y)$. Hence, in all cases $t \in (X \cup Y)$. Consequently, $I \subseteq (X \cup Y)$.
   - If $t \in X \cup Y$ then either $t \in X$ and therefore $a(t) = b(t)$ or $t \in Y$ and therefore $b(t) = c(t)$. In both cases $t \in I$. Consequently, $(X \cup Y) \subseteq I$.
   - For two sets A and B, if both $A \subseteq B$ and $B \subseteq A$ then $A = B$. Therefore, $I = X \cup Y$.

   (c) $T \setminus (X \cup Y)$ is the set of all scalene triangles (no two sides equal).

   Explanation: Let S be the set of all scalene triangles.

   - If $t \in S$ then since $a(t) \le b(t) \le c(t)$, it follows that $a(t) < b(t) < c(t)$. As a result, $t \in (T \setminus X)$ and $t \in (T \setminus Y)$, and therefore $t \in ((T \setminus X) \cap (T \setminus Y))$. By the De Morgan's laws, it follows that $t \in (T \setminus (X \cup Y))$. Consequently, $S \subseteq (T \setminus (X \cup Y))$.
   - If $t \in (T \setminus (X \cup Y))$ then $t \notin X$ and $t \notin Y$. Therefore, $a(t) < b(t)$ and $b(t) < c(t)$. As a result, $a(t) < b(t) < c(t)$, which implies that $t \in S$. Consequently, $(T \setminus (X \cup Y)) \subseteq S$.
   - For two sets A and B, if both $A \subseteq B$ and $B \subseteq A$ then $A = B$. Therefore, $S = T \setminus (X \cup Y)$.

10. For two positive integers $m < n$, let A be a set with m elements ($|A| = m$) and B a set with n elements ($|B| = n$). For each of the following sets, give upper and lower bounds on their cardinality.

An upper bound states the maximum possible elements in the set. A lower bound states the minimum possible elements in the set.

(a) $0 \leq |A \cap B| \leq m$

Explanation:

- $|A \cap B| = 0$ when A and B are disjoint sets, and therefore the lower bound is less than or equal to 0. The lower bound is exactly 0, since the cardinality of any set is greater than or equal to 0.
- $|A \cap B| = |A| = m$ when $A \subseteq B$, and therefore the upper bound is greater than or equal to m. The upper bound is exactly m, since the cardinality of the intersection of two sets is less than or equal to the cardinality of the smallest set.

(b) $n \leq |A \cup B| \leq m + n$

Explanation:

- $|A \cup B| = m + n$ when A and B are disjoint sets, and therefore the upper bound is greater than or equal to $m + n$. The upper bound is exactly $m + n$, since the cardinality of a union of two sets is less than or equal to the sum of the cardinalities of both sets.
- $|A \cup B| = |B| = n$ when $A \subseteq B$, and therefore the lower bound is less than or equal to n. The lower bound is exactly n, since the cardinality of the union of two sets is greater than or equal to the cardinality of the largest set.

(c) $0 \leq |A \setminus B| \leq m$

Explanation:

- $|A \setminus B| = |\emptyset| = 0$ when $A \subseteq B$, and therefore the lower bound is less than or equal to 0. The lower bound is exactly 0, since the cardinality of any set is greater than or equal to 0.
- $|A \setminus B| = |A| = m$ when A and B are disjoint sets, and therefore the upper bound is greater than or equal to m. The upper bound is exactly m, since the cardinality of $A \setminus B$ is less than or equal to the cardinality of A.

(d) $n - m \leq |B \setminus A| \leq n$

Explanation:

- $|B \setminus A| = n - m$ when $A \subseteq B$ because there are exactly $n - m$ elements in B that are not in A. Therefore, the lower bound is less than or equal to $n - m$. The lower bound is exactly $n - m$, since $B \setminus A$ must contain at least $n - m$ elements that are not in A.
- $|B \setminus A| = |B| = n$ when A and B are disjoint sets, and therefore the upper bound is greater than or equal to n. The upper bound is exactly n, since the cardinality of $B \setminus A$ is less than or equal to the cardinality of B.

11. There are 50 students in a class. 30 students study Math, 25 students study Computer Science, and 15 students study both Math and Computer Science. It follows that

    (a) 15 students study only Math.

    (b) 10 students study only Computer Science.

    (c) 10 students study neither Math nor Computer Science.

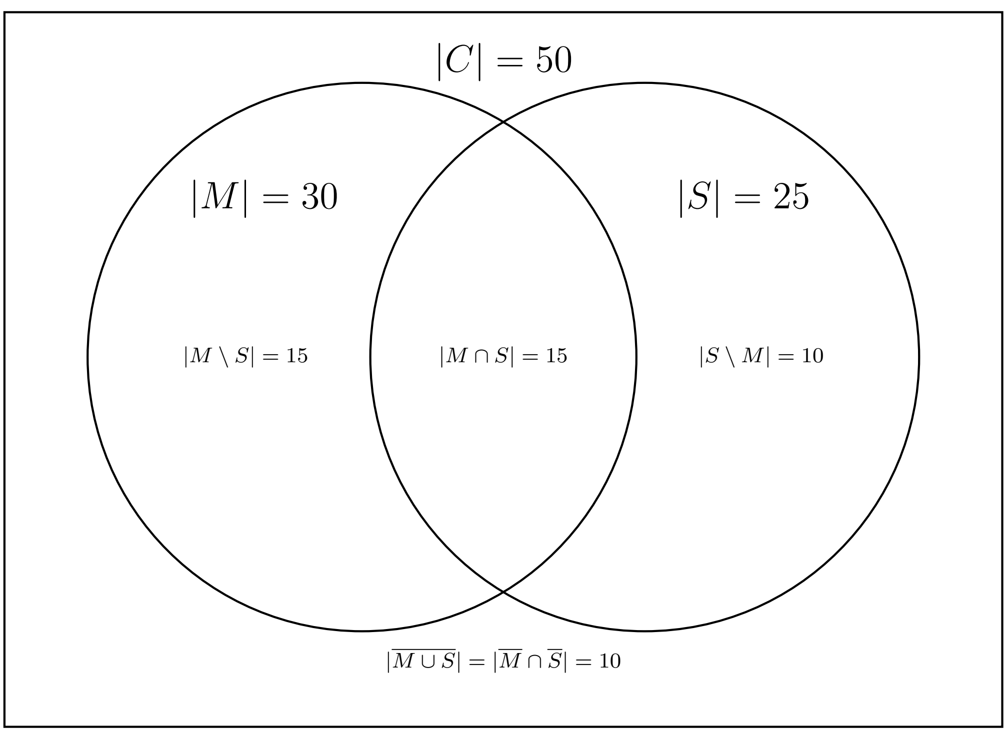
Proof: Let C be the set of all the students in the class, let M be the set of students who study Math, and let S be the set of students who study Computer Science. We know that $|C| = 50$, $|M| = 30$, $|S| = 25$, and $|M \cap S| = 15$.

(a) $M \setminus S$ is the set of students who study only Math. Since $M \setminus S = (M \setminus (M \cap S))$ and $(M \cap S) \subseteq M$, it follows that
$$|M \setminus S| = |M \setminus (M \cap S)| = |M| - |M \cap S| = 30 - 15 = 15$$

(b) $S \setminus M$ is the set of students who study only Computer Science. Since $S \setminus M = (S \setminus (S \cap M))$ and $(S \cap M) \subseteq S$, it follows that
$$|S \setminus M| = |S \setminus (S \cap M)| = |S| - |S \cap M| = 25 - 15 = 10$$

(c) $M \cup S$ is the set of students who study either Math or Computer Science or both. Therefore, $C \setminus (M \cup S)$ is the set of students who study neither Math nor Computer Science. By the inclusion-exclusion principle,
$$|M \cup S| = |M| + |S| - |M \cap S| = 30 + 25 - 15 = 40$$

Since $(M \cup S) \subseteq C$, it follows that
$$|C \setminus (M \cup S)| = |C| - |M \cup S| = 50 - 40 = 10$$



12. 36 students take the Discrete Structures class. They had 3 quizzes called X, Y, and Z.

- Surprisingly, only 1 student did not get an A on any quiz while the rest of the students got an A (*aced*) on at least one quiz.
- 21 students aced X, 19 students aced Y, and 17 students aced Z.
- 9 students aced both X and Y, 11 students aced both X and Z, and 8 students aced both Y and Z.

Definitions: Let X, Y, and Z be the sets of students who aced quizzes X, Y, and Z respectively. The problem provides the sizes of the following sets: $|X| = 21$, $|Y| = 19$, $|Z| = 17$, $|X \cap Y| = 9$, $|X \cap Z| = 11$, $|Y \cap Z| = 8$, and $|\overline{X \cup Y \cup Z}| = 1$.

Answers:

(a) How many students aced at least one quiz? $|X \cup Y \cup Z| = \mathbf{35}$.

$$|X \cup Y \cup Z| = 36 - |\overline{X \cup Y \cup Z}| = 36 - 1 = 35$$

(b) How many students aced all three quizzes? $|X \cap Y \cap Z| = \mathbf{6}$.

- The principle of inclusion and exclusion implies that

$$\begin{aligned} 35 &= |X \cup Y \cup Z| \\ &= |X| + |Y| + |Z| - |X \cap Y| - |X \cap Z| - |Y \cap Z| + |X \cap Y \cap Z| \\ &= 21 + 19 + 17 - 9 - 11 - 8 + |X \cap Y \cap Z| \\ &= 29 + |X \cap Y \cap Z| \end{aligned}$$

- This is equivalent to $|X \cap Y \cap Z| = 35 - 29 = 6$.

(c) How may students aced both X and Y but not Z? $|X \cap Y \cap \bar{Z}| = \mathbf{3}$.

$$|X \cap Y \cap \bar{Z}| = |X \cap Y| - |X \cap Y \cap Z| = 9 - 6 = 3$$

(d) How may students aced both X and Z but not Y? $|X \cap \bar{Y} \cap Z| = \mathbf{5}$.

$$|X \cap \bar{Y} \cap Z| = |X \cap Z| - |X \cap Y \cap Z| = 11 - 6 = 5$$

(e) How may students aced both Y and Z but not X? $|\bar{X} \cap Y \cap Z| = \mathbf{2}$.

$$|\bar{X} \cap Y \cap Z| = |Y \cap Z| - |X \cap Y \cap Z| = 8 - 6 = 2$$

(f) How may students aced only X? $= |X \cap \bar{Y} \cap \bar{Z}| = \mathbf{7}$.

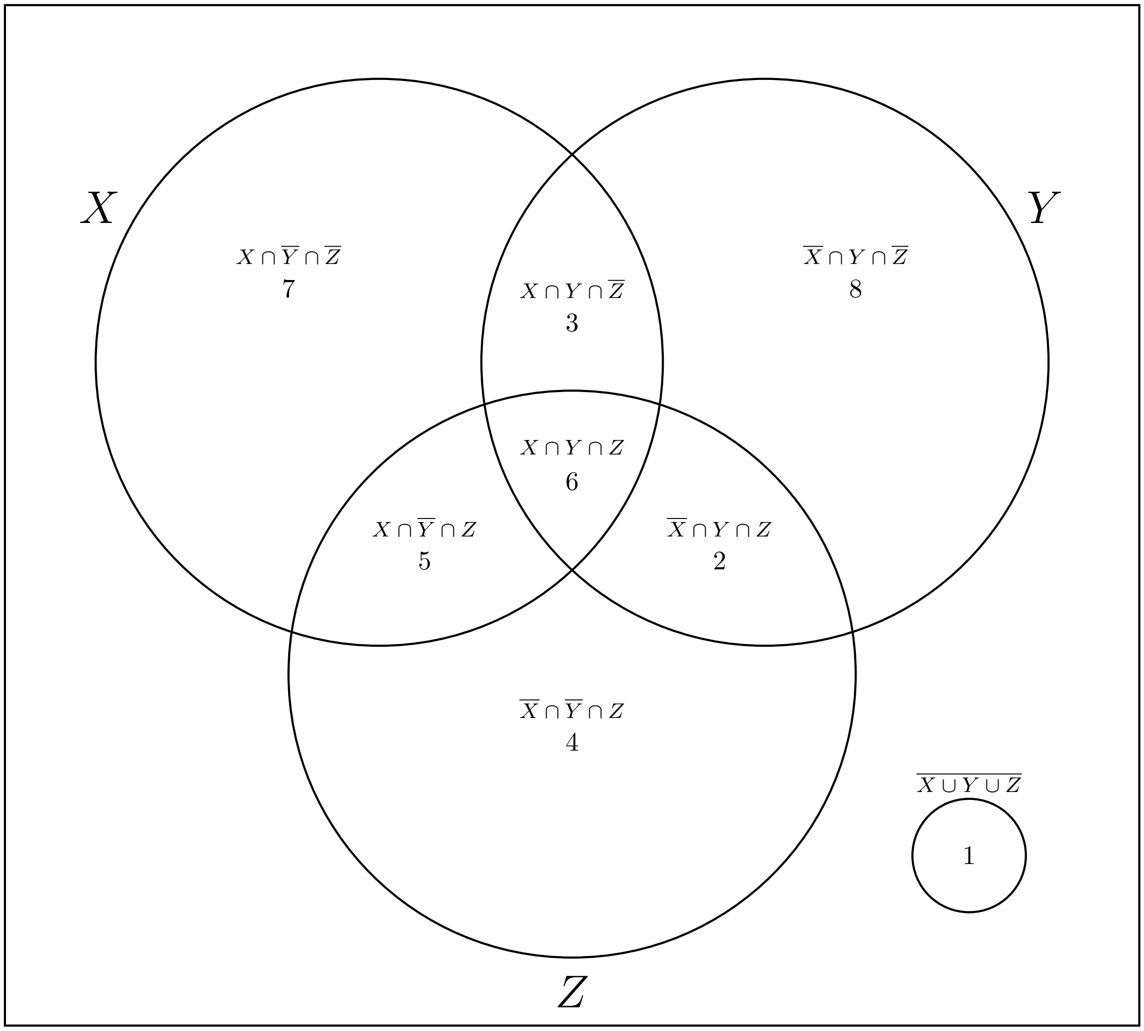
$$|X \cap \bar{Y} \cap \bar{Z}| = |X| - |X \cap Y \cap \bar{Z}| - |X \cap \bar{Y} \cap Z| - |X \cap Y \cap Z| = 21 - 3 - 5 - 6 = 7$$

(g) How may students aced only Y? $|\bar{X} \cap Y \cap \bar{Z}| = \mathbf{8}$.

$$|\bar{X} \cap Y \cap \bar{Z}| = |Y| - |X \cap Y \cap \bar{Z}| - |\bar{X} \cap Y \cap Z| - |X \cap Y \cap Z| = 19 - 3 - 2 - 6 = 8$$

(h) How may students aced only Z? $|\bar{X} \cap \bar{Y} \cap Z| = \mathbf{4}$.

$$|\bar{X} \cap \bar{Y} \cap Z| = |Z| - |X \cap \bar{Y} \cap Z| - |\bar{X} \cap Y \cap Z| - |X \cap Y \cap Z| = 17 - 5 - 2 - 6 = 4$$


X
Y
Z
$X \cap \overline{Y} \cap \overline{Z}$
7
$X \cap Y \cap \overline{Z}$
3
$\overline{X} \cap Y \cap \overline{Z}$
8
$X \cap Y \cap Z$
6
$X \cap \overline{Y} \cap Z$
5
$\overline{X} \cap Y \cap Z$
2
$\overline{X} \cap \overline{Y} \cap Z$
4
$\overline{X \cup Y \cup Z}$
1

$$36 = 6 + 3 + 2 + 5 + 7 + 8 + 4 + 1$$