The first integral of a homogeneous linear partial differential equation*

Attila Máté
Brooklyn College of the City University of New York

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1 The first integral

Consider the partial differential equation

$$P\frac{\partial u}{\partial x} + Q\frac{\partial u}{\partial y} + R\frac{\partial u}{\partial z} = 0, \tag{1}$$

where $P = P(x, y, z)$, $Q = Q(x, y, z)$, $Q = Q(x, y, z)$ are given functions. A function $u(x, y, z)$ satisfying this differential equation is also called a *first integral*. To find a first integral, one considers the behavior of u along curves $x = x(s)$, $y = y(s)$, $z = z(s)$ satisfying the differential equations

$$\frac{dx}{ds} = P, \quad \frac{dy}{ds} = Q, \quad \frac{dz}{ds} = R. \tag{2}$$

Such a curve is called a *characteristic curve* of (1) or an *integral curve* of the vector field $\langle P, Q, R\rangle$. If u is a solution of equation (1) then u must be constant along such a curve; indeed

$$\frac{du(x(s), y(s), z(s))}{ds} = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial y}\frac{dy}{ds} + \frac{\partial u}{\partial y}\frac{dy}{ds} = \frac{\partial u}{\partial x}P + \frac{\partial u}{\partial y}Q + \frac{\partial u}{\partial z}R = 0.$$

Conversely, the following is also true: if a sufficiently smooth function $u(x, y, z)$ is constant along every characteristic curve, then it is a solution of equation (1).

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1.1 Finding a first integral

To find a first integral, one uses equations (2) to find a equation $\phi(x,y,z) = c$ consistent with these equations.[1] To do this, one is not interested in the parameter s in equation (2), and writes this equation in the form (see [1, pp. 24–28])

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}. \tag{3}$$

The function $\phi(x,y,z)$ found in this way is constant along characteristic curves; if it were not, the equation $\phi(x,y,z) = c$ would not be consistent with equations (3). Hence $u = \phi(x,y,z)$ is a solution of equation (1), i.e., it is a first integral of this equation.

Example 1. Solve the differential equation

$$u_x\, x - u_y\,(xz + x) - u_z\, z = 0. \tag{4}$$

Solution. Equations (3) can be written as

$$\frac{dx}{x} = -\frac{dy}{xz + y} = -\frac{dz}{z}. \tag{5}$$

According the first and third members of these equations we have $dx/x = -dz/z$, so $\log|x| = -\log|z| + C_1$. Taking $c_1 = \pm e^{C_1}$, we have $x = c_1/z$, i.e., $xz = c_1$. That is,

$$u_1 = xz \tag{6}$$

is a first integral of the equation to be solved.

1.2 Finding a second first integral

In order to be able to find the general solution of, or the solution of an initial value problem for, equation (1), one needs to have two (functionally independent) first integrals. If one finds a first integral $u_1(x,y,z)$, one can use this first integral to find a second one by using the following procedure. With a constant c_1, use the equation $u_1(x,y,z) = c_1$ to simplify equations (3) (by eliminating one of the variables, for example), and then find an equation $\psi(x,y,z,c_1) = c_2$ consistent with the simplified equations. Then the function $u_2 = \psi(x,y,z,u_1(x,y,z))$ will also be a first integral of equation (1).

Why does this method work? The equation $u_1(x,y,z) = c_1$ describes a surface such that if it contains any point of a characteristic curve, then it contains the whole characteristic curve.[2] Then the equation $\psi(x,y,z,c_1) = c_2$, being consistent with equations (3) under the additional assumption that $u_1(x,y,z) = c_1$, describes a surface such that, if this surface contains any point of a characteristic curve the coordinates of which also satisfy the equation $u_1(x,y,z) = c_1$, then it contains the whole characteristic curve.[3] By allowing c_1 to assume arbitrary values, which amounts

[1] The word *consistent* is used in mathematics in the sense "not contradicting." What that means in this case that the equation $\phi(x,y,z) = c$ must not contradict equations (2).

[2] This statement is only true locally, namely in a small region where the solution of the characteristic equations (2) exist and is unique, given appropriate initial conditions.

[3] As will be clear from the discussion below, there there is usually only one such characteristic curve. If the surface $\psi(x,y,z,c_1) = c_2$ contains a point of a characteristic curve for which the equation $u_1(x,y,z) = c_1$ is not satisfied, then the whole characteristic curve need not lie in this surface; namely, this characteristic curve need not satisfy the equations obtained from (3) by using the equation $u_1(x,y,z) = c_1$. So $\psi(x,y,z,c_1)$ will usually *not* be a first integral of (1).

to replacing c_1 with $u_1(x, y, z)$ in the equation $\psi(x, y, z, c_1) = c_2$, these characteristic curves sweep out a surface. The equation of this surface is $\psi(x, y, z, u_1(x, y, z)) = c_2$. For this surface it will be true that if it contains any point of a characteristic curve, then it will contain the whole curve. Therefore, $u_2 = \psi(x, y, z, u_1(x, y, z))$ will be the also a first integral.

Finally, the characteristic curve for which $u_1(x, y, z) = c_1$ and $\psi(x, y, z, c_1) = c_2$ will be the intersection of the surfaces $u_1(x, y, z) = c_1$ and $u_2(x, y, z) = c_2$.[4]

Solution continued. According to equation (6), the equation $u_1 = c_1$ becomes

$$xz = c_1. \tag{7}$$

Substituting this into the first equation in (5), we obtain

$$\frac{dx}{x} = -\frac{dy}{c_1 + y},$$

i.e., $\log|x| = -\log|c_1 + y| + C_2$. Writing $c_2 = \pm e^{C_2}$, we obtain $x = c_2/(c_1 + y)$, i.e, $x(c_1 + y) = c_2$. Substituting xz for c_1 as indicated by equation (7), we obtain $x^2 z + xy = c_2$. Hence

$$u_2 = x^2 z + xy$$

is another first integral of of (4).

References

[1] E. C. Zachmanoglou and Dale W. Thoe. *Introduction to Partial Differential Equations with Applications.* Dover Publications, New York, 1986.

[4] So, if u_1 and u_2 are functionally independent, then there is only one such curve.