IRRATIONALITY OF SQUARE ROOTS[1]

Theorem. *Let n be a positive integer such that $\sqrt{n}$ is not an integer. Then $\sqrt{n}$ is irrational.*

The following proofs do not rely on the prime factorization of n. They are based on proofs that appeared at various places in the twentieth century. The first such place appears to be Carl B. Boyer's book on the history of mathematics, where the irrationality of $\sqrt{3}$ is proved. Later, the irrationality of $\sqrt{2}$ is proved along these lines by Theodor Estermann. Finally, Colin Richard Hughes, who was aware of Estermann's publication, used the method to prove the above result in its full generality. There is, however, no reason to assume that Boyer was not aware that the method is usable to prove the general result.[2]

First Proof. Assume $\sqrt{n}$ is rational. Let l be the smallest positive integer such that $\sqrt{n} = k/l$ for some integer k. Then $l\sqrt{n} = k$ and $k\sqrt{n} = (l\sqrt{n})\sqrt{n} = ln$. Let q be an integer such that $q < \sqrt{n} < q + 1$. Then

$$\sqrt{n} = \frac{k}{l} = \frac{k(\sqrt{n} - q)}{l(\sqrt{n} - q)} = \frac{k\sqrt{n} - kq}{l\sqrt{n} - lq} = \frac{ln - kq}{k - lq}.$$

On the right-hand side $\sqrt{n}$ is represented as the ratio of two integers. Since $k - lq = l(\sqrt{n} - q)$, and $0 < \sqrt{n} - q < 1$, we have

$$0 < k - lq < l,$$

which contradicts the choice of l, according to which l is the least positive integer for which $\sqrt{n} = k/l$. This contradiction shows that $\sqrt{n}$ is irrational. $\square$

The second proof is essentially the same as the first proof, but it is explained somewhat differently.

Second Proof. Assume, on the contrary, that $\sqrt{n}$ is rational. Then there are positive integers k and l such that $\sqrt{n} = k/l$, i.e., such that $l\sqrt{n} = k$. Assume l is the smallest integer for which an integer k satisfying this equation exists. Let q and r be integers such that $k = ql + r$ and $0 \le r < l$. We cannot have $r = 0$ here since $r = 0$ would mean that $lq = k$, which, together with the equation $l\sqrt{n} = k$ would mean that $q = \sqrt{n}$, whereas we assumed that $\sqrt{n}$ is not an integer. Then we have $k\sqrt{n} = (l\sqrt{n})\sqrt{n} = ln$, and so

$$r\sqrt{n} = (k - ql)\sqrt{n} = k\sqrt{n} - ql\sqrt{n} = ln - qk.$$

Hence $r\sqrt{n}$ is an integer. This is, however, a contradiction, since $0 < r < l$ and l is the smallest positive integer for which $l\sqrt{n}$ is an integer. This contradiction proves that $\sqrt{n}$ is not rational. $\square$

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[1] Notes for courses at Brooklyn College of CUNY. Attila Máté, October 30, 2009. Revised February 16, 2014.

[2] See Carl B. Boyer, A History of Mathematics, Wiley International Edition, John Wiley & Sons, Inc, New York–London–Sydney, 1968, `https://archive.org/details/AHistoryOfMathematics`, p. 387, Theodor Estermann, *The irrationality of* $\sqrt{2}$, Math. Gaz., Vol. 59, No. 408 (1975), p. 110, `http://www.jstor.org/stab le/3616647`, and Colin Richard Hughes, *Irrational roots*, Math. Gaz., Vol. 83, No. 498 (1999), pp. 502–503. `http://www.jstor .org/stable/3620971`. Boyer also describes geometric proofs of the irrationality of $\sqrt{2}$ and $\sqrt{5}$ that do not rely on prime factorization, either; see pp. 80–81 in his book quoted above.

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