THE REMAINDER TERM IN TAYLOR'S FORMULA[1]

In Taylor's formula, a function $f(x)$ is approximated by a polynomial

$$\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

The goodness of this approximation can be measured by the remainder term $R_n(x,a)$, defined as

$$R_n(x,a) \overset{def}{=} f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

To estimate $R_n(x,a)$, we need the following lemma.

Lemma. *Let $n \ge 1$ be an integer. Let U be an open interval in $\mathbb{R}$ and let $f : U \to \mathbb{R}$ be a function that is $n+1$ times differentiable. Given any $b \in U$, we have*

$$\frac{d}{dx}R_n(b,x) = -\frac{f^{(n+1)}(x)(b-x)^n}{n!} \tag{1}$$

for every $x \in U$.

Proof. We have

$$R_n(b,x) = f(b) - \sum_{k=0}^{n} f^{(k)}(x)\frac{(b-x)^k}{k!} = f(b) - f(x) - \sum_{k=1}^{n} f^{(k)}(x)\frac{(b-x)^k}{k!}.$$

We separated out the term for $k=0$ since we are going to use the product rule for differentiation, and the term for $k=0$ involves no product. We have

$$\begin{aligned}
\frac{d}{dx}R_n(b,x) &= \frac{d}{dx}f(b) - \frac{d}{dx}f(x) \\
&\quad - \sum_{k=1}^{n}\left(\frac{df^{(k)}(x)}{dx}\frac{(b-x)^k}{k!} + f^{(k)}(x)\frac{d}{dx}\frac{(b-x)^k}{k!}\right) \\
&= -f'(x) - \sum_{k=1}^{n}\left(f^{(k+1)}(x)\frac{(b-x)^k}{k!} + f^{(k)}(x)\frac{-k(b-x)^{k-1}}{k!}\right) \\
&= -f'(x) - \sum_{k=1}^{n}\left(f^{(k+1)}(x)\frac{(b-x)^k}{k!} - f^{(k)}(x)\frac{(b-x)^{k-1}}{(k-1)!}\right).
\end{aligned}$$

Writing

$$A_k = f^{(k+1)}(x)\frac{(b-x)^k}{k!}$$

for k with $0 \le k \le n$, the sum (i.e., the expression described by $\sum_{k=1}^{n}$) on the right-hand side equals

$$\begin{aligned}
\sum_{k=1}^{n}(A_k - A_{k-1}) &= (A_1 - A_0) + (A_2 - A_1) + \dots (A_n - A_{n-1}) \\
&= A_n - A_0 = f^{(n+1)}(x)\frac{(b-x)^n}{n!} - f'(x).
\end{aligned}$$

[1] Notes for Course Mathematics 1206 Attila Máté, Brooklyn College of CUNY. April 26, 2012.

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Substituting this in the above equation, we obtain

$$\frac{d}{dx}R_n(b,x) = -f'(x) - \left(f^{(n+1)}(x)\frac{(b-x)^n}{n!} - f'(x)\right) = -f^{(n+1)}(x)\frac{(b-x)^n}{n!},$$

as we wanted to show.

Corollary 1. *Let $n \geq 1$ be an integer. Let U be an open interval in $\mathbb{R}$ and let $f : U \to \mathbb{R}$ be a function that is $n+1$ times differentiable. For any any $a, b \in U$ with $a \neq b$, there is a $\xi \in (a,b)$ (if $a < b$) or $\xi \in (b,a)$ (if $a > b$) such that*

$$R_n(b,a) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(b-a)^{n+1}. \tag{2}$$

Proof. For the sake of simplicity, we will assume that $a < b$.[2] We have $b - a \neq 0$, and so the equation

$$R_n(b,a) = K \cdot \frac{(b-a)^{n+1}}{(n+1)!} \tag{3}$$

can be solved for K. Let K be the real number for which this equation is satisfied, and write

$$\phi(x) = R_n(b,x) - K \cdot \frac{(b-x)^{n+1}}{(n+1)!}$$

Then ϕ is differentiable in U; as differentiability implies continuity, it follows that f is continuous on the interval $[a,b]$ and differentiable in (a,b).[3] As $\phi(a) = 0$ by the choice of K and $\phi(b) = 0$ trivially, we can use Rolle's Theorem to obtain the existence of a $\xi \in (a,b)$ such that $\phi'(\xi) = 0$. Using (1), we can see that

$$0 = \phi'(\xi) = -\frac{f^{(n+1)}(\xi)(b-\xi)^n}{n!} - K \cdot \frac{-(n+1)(b-\xi)^n}{(n+1)!}.$$

Noting that $\frac{(n+1)}{(n+1)!} = \frac{1}{n!}$ and keeping in mind that $\xi \neq b$, we obtain $K = f^{n+1}(\xi)$ from here. Thus the result follows from (3).

Note. The above argument can be carried out in somewhat more generality. Let $g(x)$ be a function that is differentiable on (a,b), $g'(x) \neq 0$ for $x \in (a,b)$, and $g(b) = 0$. Note that, by the Mean-Value Theorem,

$$-\frac{g(a)}{b-a} = \frac{g(b) - g(a)}{b-a} = g'(\eta)$$

for some $\eta \in (a,b)$. Since we have $g'(\eta) \neq 0$ by our assumptions, it follows that $g(a) \neq 0$. Instead of (3), determine K now such that

$$R_n(b,a) = Kg(a). \tag{4}$$

As $g(a) \neq 0$, it is possible to find such a K. Write

$$\phi(x) = R_n(b,x) - Kg(x).$$

We have $\phi(a) = 0$ by the choice of K. We have $\phi(b) = 0$ since $R_n(b,b) = 0$ and $g(b) = 0$ (the latter by our assumptions). As ϕ is differentiable on (a,b), there is a $\xi \in (a,b)$ such that $\phi'(\xi) = 0$. Thus, by (1) we have

$$0 = \phi'(\xi) = -\frac{f^{(n+1)}(\xi)(b-\xi)^n}{n!} - Kg'(\xi).$$

As $g'(\xi) \neq 0$ by our assumptions, we can determine K from this equation. Substituting the value of K so obtained into (4), we can see that

$$R_n(b,a) = -\frac{f^{(n+1)}(\xi)(b-\xi)^n}{n!} \cdot \frac{g(a)}{g'(\xi)}.$$

Note that in the argument we again assumed that $a < b$, but this was unessential. Further, note that the function g can depend on a and b. We can restate the result just obtained in the following

[2]This assumption is never really used except that it helps us avoid circumlocutions such as $\xi \in (a,b)$ if $a < b$ or $\xi \in (b,a)$ if $b < a$.

[3]Naturally, ϕ is differentiable also at a and b, but this is not needed for the rest of the argument.

Corollary 2. *Let $n \geq 1$ be an integer. Let U be an open interval in $\mathbb{R}$ and let $f : U \to \mathbb{R}$ be a function that is $n+1$ times differentiable. For any any $a, b \in U$ with $a < b$, let $g_{a,b}(x)$ be a function such that $g_{a,b}$ is continuous on $[a, b]$ and differentiable on (a, b). Assume, further, that $g_{a,b}(b) = 0$ and $g'_{a,b}(x) \neq 0$ for $x \in (a, b)$. Then there is a $\xi \in (a, b)$ such that*

$$R_n(b, a) = -\frac{f^{(n+1)}(\xi)(b-\xi)^n}{n!} \cdot \frac{g_{a,b}(a)}{g'_{a,b}(\xi)}.$$

If $b < a$, then the same result holds, except that one needs to write (b, a) instead of (a, b) and $[b, a]$ instead of $[a, b]$ for the intervals mentioned (but the roles of a and b should not otherwise be interchanged).

This result is given in [B]; see also [M]. The Wikipedia entry [Wiki] also discusses the result (especially under the subheading *Mean value theorem*), without giving attributions. Taking $g_{a,b}(x) = (b-x)^r$ for an integer r with $0 < r \leq n+1$, we obtain that

$$R_n(b, a) = \frac{f^{(n+1)}(\xi)}{rn!}(b-\xi)^{n-r+1}(b-a)^r.$$

This is called the Roche–Schlömilch Remainder Term of the Taylor Series. Here ξ is some number in the interval (a, b) or (b, a); it is important to realize that the value of ξ depends on r. Taking $r = n+1$ here, we get formula (2); this is called Lagrange's Remainder Term of the Taylor Series. Taking $r = 1$, we obtain

$$R_n(b, a) = \frac{f^{(n+1)}(\xi)}{n!}(b-\xi)^n(b-a);$$

this is called Cauchy's Remainder Term of the Taylor Series.

The different forms of the remainder term of Taylor's series are useful in different situations. For example, Lagrange's Remainder Term is convenient for establishing the convergence of the Taylor series of e^x, $\sin x$, and $\cos x$ on the whole real line, but it is not suited to establish the convergence of the Taylor series of $(1+x)^\alpha$, where α is an arbitrary real number. The Taylor series of this last function is convergent on the interval $(-1, 1)$, and on this interval it does converge to the function $(1+x)^\alpha$ (this series is called the *Binomial Series*). This can be established by using Cauchy's Remainder Term.