# Telescoping sums[*]

Attila Máté

Brooklyn College of the City University of New York

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## 1 Telescoping sums

The following two equations are easy to show. Let $a_k$ be a real number for all integers $k \geq 0$. Then

$$\text{(1)} \qquad \sum_{k=1}^{n}(a_k - a_{k-1}) = a_n - a_0$$

for every integer $n \geq 0$. Indeed, we have

$$\sum_{k=1}^{n}(a_k - a_{k-1}) = (a_1 - a_0) + (a_2 - a_1) + (a_3 - a_2) + (a_4 - a_3) + \ldots$$
$$+ (a_{n-2} - a_{n-3}) + (a_{n-1} - a_{n-2}) + (a_n - a_{n-1}) = a_n - a_0;$$

the last equation holds because of cancelations. Similarly, let $a_k$ be a real number for all integers $k \geq 1$. Then

$$\text{(2)} \qquad \sum_{k=1}^{n}(a_k - a_{k+1}) = a_1 - a_{n+1}$$

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for every integer $n \geq 0$. Note that for $n \geq 1$ the equations are true since all terms not shown on the right-hand side cancel; for $n = 0$ they are true since the sums on the left are empty, and empty sums are 0 by convention, while the right-hand sides are also 0. Such sums are called telescoping or collapsing sums. The reason for the name "telescoping" is that some telescopes are constructed of several tubes of different diameters that can be collapsed or pushed into one another for compact storing of the telescope.

## 2 Telescoping sums for polynomials and their reciprocals

### 2.1 Sums for polynomials: an example for formula (1)

As an example for applications of the first equation, consider

$$(3) \qquad a_k = \prod_{j=0}^{l}(k+j) \qquad (l \geq 0).$$

Then, for $k \geq 1$ we have

$$a_k - a_{k-1} = \prod_{j=0}^{l}(k+j) - \prod_{j=0}^{l}(k-1+j) = (k+l)\prod_{j=0}^{l-1}(k+j) - (k-1)\prod_{j=1}^{l}(k-1+j)$$

$$= (k+l)\prod_{j=0}^{l-1}(k+j) - (k-1)\prod_{j=0}^{l-1}(k+j) = \big((k+l)-(k-1)\big)\prod_{j=0}^{l-1}(k+j)$$

$$= (l+1)\prod_{j=0}^{l-1}(k+j).$$

Observe that this calculation is valid even in case $l = 0$, since the empty product is taken to equal 1 by convention. Noting that $a_0 = 0$, it follows from (1) that

$$\sum_{k=1}^{n}(l+1)\prod_{j=0}^{l-1}(k+j) = \prod_{j=0}^{l}(n+j);$$

that is, dividing both sides by $l + 1$, we obtain

$$(4) \qquad \sum_{k=1}^{n}\prod_{j=0}^{l-1}(k+j) = \frac{1}{l+1}\prod_{j=0}^{l}(n+j).$$

For $l = 0$ this gives

$$\sum_{k=1}^{n}1 = n.$$

For $l = 1$ it gives

$$\sum_{k=1}^{n}k = \frac{n(n+1)}{2}.$$

For $l = 2$ it gives
$$\sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}.$$

For $l = 3$ it gives
$$\sum_{k=1}^{n} k(k+1)(k+2) = \frac{n(n+1)(n+2)(n+3)}{4}.$$

As an application of these formulas, we have

$$\sum_{k=1}^{n} k^2 = \sum_{k=1}^{n} \big(k(k+1) - k\big) = \sum_{k=1}^{n} k(k+1) - \sum_{k=1}^{n} k = \frac{n(n+1)(n+2)}{3} - \frac{n(n+1)}{2}$$

$$= \frac{2n(n+1)(n+2) - 3n(n+1)}{6} = \frac{n(n+1)\big(2(n+2) - 3\big)}{6} = \frac{n(n+1)(2n+1)}{6}.$$

## 2.2   Sums for reciprocals of polynomials: an example for formula (2)

As an example for the application of the second equation, consider

(5)
$$a_k = \frac{1}{\prod_{j=0}^{l}(k+j)} \qquad (l \geq 0).$$

Then, for $k \geq 1$ we have

$$a_k - a_{k+1} = \frac{1}{\prod_{j=0}^{l}(k+j)} - \frac{1}{\prod_{j=0}^{l}(k+1+j)} = \frac{1}{\prod_{j=0}^{l}(k+j)} - \frac{1}{\prod_{j=1}^{l+1}(k+j)}$$

$$= \frac{k+l+1}{(k+l+1)\prod_{j=0}^{l}(k+j)} - \frac{k}{k\prod_{j=1}^{l+1}(k+j)} = \frac{k+l+1}{\prod_{j=0}^{l+1}(k+j)} - \frac{k}{\prod_{j=0}^{l+1}(k+j)}$$

$$= \frac{(k+l+1) - k}{\prod_{j=0}^{l+1}(k+j)} = \frac{l+1}{\prod_{j=0}^{l+1}(k+j)}.$$

Thus, by equation (2) we have

$$\sum_{k=1}^{n} \frac{l+1}{\prod_{j=0}^{l+1}(k+j)} = \frac{1}{\prod_{j=0}^{l}(1+j)} - \frac{1}{\prod_{j=0}^{l}(n+1+j)}$$

$$= \frac{1}{\prod_{j=1}^{l+1} j} - \frac{1}{\prod_{j=1}^{l+1}(n+j)} = \frac{1}{(l+1)!} - \frac{1}{\prod_{j=1}^{l+1}(n+j)}$$

Dividing both sides by $l+1$, we obtain

(6)
$$\sum_{k=1}^{n} \frac{1}{\prod_{j=0}^{l+1}(k+j)} = \frac{1}{l+1} \left( \frac{1}{(l+1)!} - \frac{1}{\prod_{j=1}^{l+1}(n+j)} \right)$$

For $l = 0$ this gives
$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = 1 - \frac{1}{n+1}.$$

For $l = 1$ it gives

$$\sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)} = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{(n+1)(n+2)}\right).$$

For $l = 2$ it gives

$$\sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)(k+3)} = \frac{1}{3}\left(\frac{1}{6} - \frac{1}{(n+1)(n+2)(n+3)}\right).$$

For $l = 3$ it gives

$$\sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)(k+3)(k+4)} = \frac{1}{4}\left(\frac{1}{24} - \frac{1}{(n+1)(n+2)(n+3)(n+4)}\right).$$

## 2.3   Sums of binomial coefficients

The binomial coefficient $\binom{\alpha}{k}$ for any integer $k \geq 0$ and any real $\alpha$ is defined as

(7)
$$\binom{\alpha}{k} \stackrel{def}{=} \prod_{j=0}^{k-1} \frac{\alpha - j}{k - j} = \frac{1}{k!}\prod_{j=0}^{k-1}(\alpha - j).$$

From here it is easy to conclude that

(8)
$$\binom{\alpha}{k+1} = \frac{\alpha - k}{k+1}\binom{\alpha}{k} \quad \text{and} \quad \binom{\alpha+1}{k+1} = \frac{\alpha+1}{k+1}\binom{\alpha}{k}$$

Hence

(9)
$$\binom{\alpha}{k} + \binom{\alpha}{k+1} = \left(1 + \frac{\alpha - k}{k+1}\right)\binom{\alpha}{k} = \frac{\alpha+1}{k+1}\binom{\alpha}{k} = \binom{\alpha+1}{k+1}$$

for any real $\alpha$ and any integer $k \geq 0$,

   If $\alpha$ is a positive integer, then equation (7) agrees with the well-known definition of the binomial coefficients. Even if $\alpha$ is not an integer, the definition is used in describing the binomial series

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k}x^k \qquad (|x| < 1).$$

In case $\alpha$ is a positive integer then $\binom{\alpha}{k} = 0$ for $k > \alpha$, since the factor $\alpha - j$ is zero in the numerator on the right-hand side of (7) for $j = \alpha$, and the above equation becomes the binomial formula. For positive integers $\alpha$, equation (9) is the well-known property used in building the Pascal triangle, saying that each entry in the Pascal triangle is the sum of the two neighboring entries above it.

   Given positive integers $k$ and $m$ with $m \geq k$ and a real $\alpha$, using equation (9) we can see that

$$\sum_{n=k}^{m} \binom{\alpha+n}{k} = \sum_{n=k}^{m}\left(\binom{\alpha+n+1}{k+1} - \binom{\alpha+n}{k+1}\right)$$
$$= \binom{\alpha+m+1}{k+1} - \binom{\alpha+k}{k+1}.$$

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In case $\alpha = 0$, this becomes

$$\sum_{n=k}^{m} \binom{n}{k} = \binom{m+1}{k+1} - \binom{k}{k+1} = \binom{m+1}{k+1},$$

where the last equation holds since $\binom{k}{k+1} = 0$. This equation is easily shown to be equivalent to equation (4) after some change of notation.

# 3 Trigonometric telescoping sums

## 3.1 A sum Archimedes considered

The sum

$$(10) \qquad\qquad S = \sum_{k=1}^{n-1} \sin \frac{k\theta}{n}$$

can be evaluated using the trigonometric identities

$$(11) \qquad\qquad 2 \sin x \sin y = \cos(x - y) - \cos(x + y).$$

This identity is an easy consequence of the addition and the subtraction formulas for cosine:

$$(12) \qquad\qquad \begin{aligned} \cos(x + y) &= \cos x \cos y - \sin x \sin y, \\ \cos(x - y) &= \cos x \cos y + \sin x \sin y. \end{aligned}$$

The second one of these is not really a separate formula, since it is a consequence of the first one by replacing $y$ with $-y$ and noting that $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$:

$$\cos(x - y) = \cos(x + (-y)) = \cos x \cos(-y) - \sin x \sin(-y) = \cos x \cos y + \sin x \sin y.$$

Multiplying equation (10) by $2 \sin(\theta/2)$ and using identity (11), we obtain

$$2S \sin \frac{\theta}{2n} = \sum_{k=1}^{n-1} 2 \sin \frac{k\theta}{n} \sin \frac{\theta}{2n} = \sum_{k=1}^{n-1} \left( \cos \frac{(2k-1)\theta}{2n} - \cos \frac{(2k+1)\theta}{2n} \right)$$

Writing

$$a_k = \cos \frac{(2k-1)\theta}{2n},$$

the sum on the right hand side can be written as

$$\sum_{k=1}^{n-1} (a_k - a_{k+1}) = a_1 - a_n.$$

where the equation holds according to (2) with $n - 1$ replacing $n$. Thus

$$(13) \qquad\qquad 2S \sin \frac{\theta}{2n} = \cos \frac{\theta}{2n} - \cos \frac{(2n-1)\theta}{2n}.$$

In other words

$$S = \frac{\cos \frac{\theta}{2n} - \cos \frac{(2n-1)\theta}{2n}}{2 \sin \frac{\theta}{2n}}.$$

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## 3.2 Archimedes's sum and the surface area of the sphere segment

Given the unit circle $x^2 + y^2 = 1$, using polar coordinates, the surface area of a segment of the sphere obtained by rotation about the $x$-axis the arc of this circle between the angles $\theta = 0$ and $\theta = \theta_0$ ($0 \le \theta_0 \le \pi$) can be expressed as the integral

$$
(14) \qquad \int_0^{\theta_0} 2\pi \sin\theta \, d\theta = 2\pi(1 - \cos\theta_0).
$$

This integral was evaluated by Archimedes. Using geometry, he established an identity that can be expressed in terms of trigonometry as the identity

$$
(15) \qquad 2\sum_{k=1}^{n-1} \sin\frac{k\theta}{n} \sin\frac{\theta}{2n} + \sin\theta \sin\frac{\theta}{2n} = \cos\frac{\theta}{2n} - \cos\theta \cos\frac{\theta}{2n}.
$$

Moving the term after the sum to the right-hand side and using equation (13), this identity can be seen to be equivalent to

$$
\cos\frac{\theta}{2n} - \cos\frac{(2n-1)\theta}{2n} = \cos\frac{\theta}{2n} - \left(\cos\theta \cos\frac{\theta}{2n} + \sin\theta \sin\frac{\theta}{2n}\right).
$$

Applying the subtraction formula for cosine in (12) to the expression in parentheses on the right-hand side, the two sides become identical, showing that Archimedes's identity (15) is indeed valid.

We have

$$
\int_0^\theta \sin x \, dx = \lim_{n\to\infty} \sum_{k=1}^n \left(\sin\frac{k\theta}{n}\right)\frac{\theta}{n} = \lim_{n\to\infty} 2\sum_{k=1}^n \frac{\frac{\theta}{2n}}{\sin\frac{\theta}{2n}} \sin\frac{k\theta}{n} \sin\frac{\theta}{2n}
$$

$$
= \lim_{n\to\infty} 2\sum_{k=1}^n \sin\frac{k\theta}{n} \sin\frac{\theta}{2n} = \lim_{n\to\infty} \left(2\sum_{k=1}^{n-1} \sin\frac{k\theta}{n} \sin\frac{\theta}{2n} + \sin\theta \sin\frac{\theta}{2n}\right);
$$

The second equality here holds in view of the equation $\lim_{x\to0}(x/\sin x) = 1$ with $\theta/(2n)$ replacing $x$, and the third one follows since $\lim_{n\to\infty} \sin(\theta/(2n)) = 0$. According to equation (15), the right-hand side here equals

$$
\lim_{n\to\infty} \left(\cos\frac{\theta}{2n} - \cos\theta \cos\frac{\theta}{2n}\right) = 1 - \cos\theta \, ;
$$

the equation here holds since $\lim_{x\to0} \cos x = 1$, with $\theta/(2n)$ replacing $x$. Hence

$$
\int_0^\theta \sin x \, dx = 1 - \cos\theta,
$$

which, with a change of notation, is equivalent to equation (14).